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AERODYNAMIC SYMMETRY OF PROJECTILES[1]

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1. Introduction. The theory of the motion of a projectile is usually based on the assumption that the projectile possesses complete rotational symmetry, in the sense that rotation through any angle about the axis of symmetry leaves the projectile as a whole unchanged in outward form and in internal mass distribution. This assumption of symmetry is suitable in the case of an ordinary shell, but is not suitable in the case of a finned projectile which possesses symmetry of a more restricted type. The purpose of the present paper is to investigate the effect of various types of rotational and reflectional symmetry on the aerodynamic force system acting on a projectile. According to the basic aerodynamic hypothesis[2] the aerodynamic force system depends on the instantaneous motion of the projectile. The form of this functional dependence is restricted by the symmetry of the projectile. Our purpose is to investigate this restriction mathematically.

The instantaneous motion of a projectile is described by choosing some base-point O, fixed in the projectile, and specifying the velocity ($\mathbf{u}$) of O and the angular velocity ($\boldsymbol{\omega}$) of the projectile. The aerodynamic force system, exerted by the air on the projectile, is described by the equipollent system consisting of a force ($\mathbf{F}$) acting at O, together with a couple ($\mathbf{G}$). Then the vectors $\mathbf{F}$ and $\mathbf{G}$ are functions of the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$. They are also functions of the air density (ρ) and the local sound velocity (c), but these functional dependences will be suppressed as a matter of notational convenience. Choosing any rectangular coordinate axes $Ox_1x_2x_3$, we may express the functional dependence of force system on motion by the six equations

$$F_m = f_m(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3),$$
$$G_m = g_m(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3), \qquad (m = 1, 2, 3), \tag{1.1}$$

u_m, ω_m, F_m, G_m being components along Ox_m.

2. Types of symmetry. Let us consider projectiles possessing one or both of the following types of symmetry (Fig. 1):

(*i*) n-gonal rotational symmetry about an axis A;

(*ii*) reflectional symmetry in a plane P.

Symmetries may be described in terms of "covering operations" which leave the projectile as a whole unchanged. We shall be concerned only with the aerodynamic force

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system and not with the dynamics of the projectile. This means that we are interested in its external form, but not in its distribution of mass. Consequently all remarks about symmetry are to be interpreted relative to the external form only. The covering operation for n-gonal rotational symmetry about an axis A is a rotation about A through an angle α where $\alpha = 2\pi/n$, n being an integer greater than unity. (An ordinary bomb with four fins possesses tetragonal symmetry, $n = 4$.) For certain formal purposes, it is convenient to include the case $n = 1$, although this actually implies no symmetry. The covering operation for reflectional symmetry in a plane P is a reflection in P in the ordinary geometrical sense, i.e. each particle of the projectile is replaced by another particle at a position such that the line joining the two positions is bisected orthogonally by P. The covering operation for n-gonal symmetry may be carried out physically; that for reflectional symmetry cannot.

However, we may view the question of symmetry in a different, but equivalent, way. We may apply the covering operations, not to the projectile, but to a set of axes. Consider first the case of n-gonal symmetry about an axis A. Choose a set of rectangular axes $Ox_1x_2x_3$ with origin on A. Apply the covering operation to these axes, leaving the body unmoved. Let the new axes be $Ox_1'x_2'x_3'$. The two sets of axes are *equivalent* in the following sense: Any general statement about the connection between aerodynamic force system and motion must have the same *form* no matter which axes are used. Otherwise put, it is impossible for an experimenter to determine, by aerodynamic measurements, which of the two equivalent sets of axes he is using. This means that the six aerodynamic functions in (1.1) are the same for two sets of equivalent axes. If primes indicate components along $Ox_1'x_2'x_3'$, we must have

$$\begin{aligned} F_m' &= f_m(u_1', u_2', u_3', \omega_1', \omega_2', \omega_3'), \\ G_m' &= g_m(u_1', u_2', u_3', \omega_1', \omega_2', \omega_3'), \qquad (m = 1, 2, 3), \end{aligned} \tag{2.1}$$

f_m and g_m being *the same functions* in (1.1) and (2.1).

The same remarks apply in the case of reflectional symmetry in a plane P. In this case, we choose the origin O on P, and reflect $Ox_1x_2x_3$ in P to obtain $Ox_1'x_2'x_3'$. However some care must be exercised in this case because the orientation of the axes is changed. This introduces no difficulty in the case of the polar vectors **u** and **F**, but must be taken into consideration in the case of the axial vectors **ω** and **G**. This point will be discussed below in the appropriate place.

On account of the ease with which the notation of complex variable lends itself to the treatment of rotations in a plane, we find it convenient to use the following notation:

$$u = u_1 + iu_2, \qquad \omega = \omega_1 + i\omega_2, \qquad F = F_1 + iF_2, \qquad G = G_1 + iG_2, \tag{2.2}$$

with similar definitions for the primed quantities. Let us rewrite (1.1) and (2.1) in the form

$$\begin{aligned} F &= f(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \qquad F_3 = f_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ G &= g(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \qquad G_3 = g_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \end{aligned} \tag{2.3}$$

$$\begin{aligned} F' &= f(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3'), \qquad F_3' = f_3(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3') \\ G' &= g(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3'), \qquad G_3' = g_3(u', \bar{u}', u_3', \omega', \bar{\omega}', \omega_3'), \end{aligned} \tag{2.4}$$

where the bars denote complex conjugates. Here the functions f and g are complex functions of complex arguments, and f_3 and g_3 are real functions of complex arguments. Since we have changed the arguments in passing from (1.1) and (2.1) to (2.3) and (2.4), the functional forms f_3 and g_3 are different in the two sets of equations. No confusion will arise from this, because we shall not refer to (1.1) or (2.1) again.

3. Consequences of n-gonal rotational symmetry. We suppose that the projectile has n-gonal rotational symmetry about an axis A. We choose the origin O on A, the axis Ox_3 along A, and the other two axes in the plane perpendicular to A, but otherwise arbitrary except for the condition that they must of course be at right angles. We consider some state of motion and the corresponding aerodynamic force system. These are described respectively by the complex and real components $(u, u_3, \omega, \omega_3)$ and (F, F_3, G, G_3) on the axes $Ox_1x_2x_3$. Now apply to the axes the covering operation, i.e. a rotation about Ox_3 through an angle $\alpha = 2\pi/n$. *The same motion and the corresponding force system* may also be described by the complex and real components $(u', u_3', \omega', \omega_3')$, (F', F_3', G', G_3') on the new axes $Ox_1'x_2'x_3'$. The transformation from one set of components to the other is easily seen to be as follows:

$$\begin{aligned} u' &= ue^{-i\alpha}, \quad & u_3' &= u_3, \quad & \omega' &= \omega e^{-i\alpha}, \quad & \omega_3' &= \omega_3, \\ F' &= Fe^{-i\alpha}, \quad & F_3' &= F_3, \quad & G' &= Ge^{-i\alpha}, \quad & G_3' &= G_3. \end{aligned} \tag{3.1}$$

If we substitute these in (2.4) and then compare with (2.3), we get the following equations, which must be satisfied for arbitrary values of $(u, u_3, \omega, \omega_3)$:

$$\begin{aligned} e^{i\alpha} f(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega} e^{i\alpha}, \omega_3) &= f(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ e^{i\alpha} g(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega} e^{i\alpha}, \omega_3) &= g(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ f_3(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega} e^{i\alpha}, \omega_3) &= f_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3), \\ g_3(ue^{-i\alpha}, \bar{u}\, e^{i\alpha}, u_3, \omega e^{-i\alpha}, \bar{\omega} e^{i\alpha}, \omega_3) &= g_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3). \end{aligned} \tag{3.2}$$

We now make an important assumption, namely, that the functions f, f_3, g, g_3 may be expanded in power series in the transverse components $u_1, u_2, \omega_1, \omega_2$, or, equivalently, in power series in the complex quantities $u, \bar{u}, \omega, \bar{\omega}$. We shall write these power series in the form

$$\begin{aligned} F &= f(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} F_{pqrs}(u_3, \omega_3) u^p \bar{u}^q \omega^r \bar{\omega}^s, \\ G &= g(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} G_{pqrs}(u_3, \omega_3) u^p \bar{u}^q \omega^r \bar{\omega}^s, \\ F_3 &= f_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} F^{(3)}_{pqrs}(u_3, \omega_3) u^p \bar{u}^q \omega^r \bar{\omega}^s, \\ G_3 &= g_3(u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3) = \sum_{pqrs} G^{(3)}_{pqrs}(u_3, \omega_3) u^p \bar{u}^q \omega^r \bar{\omega}^s, \end{aligned} \tag{3.3}$$

the summations covering the ranges $p, q, r, s = 0, 1, 2, \cdots$. Here the functions

$$F_{pqrs}(u_3, \omega_3),\ G_{pqrs}(u_3, \omega_3),\ F^{(3)}_{pqrs}(u_3, \omega_3),\ G^{(3)}_{pqrs}(u_3, \omega_3)$$

are the same functions for equivalent axes.

We note that the coefficients $F^{(3)}_{pqrs}$ and $G^{(3)}_{pqrs}$ are in general complex, and certain relations must be satisfied by them in order that f_3 and g_3 may be real. We have $f_3 = \bar{f}_3$, and so

$$\sum_{pqrs} F^{(3)}_{pqrs}(u_3 , \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s = \sum_{pqrs} \bar{F}^{(3)}_{pqrs}(u_3 , \omega_3)\bar{u}^p u^q\bar{\omega}^r\omega^s$$
$$= \sum_{pqrs} \bar{F}^{(3)}_{qpsr}(u_3 , \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s. \tag{3.4}$$

This holds for arbitrary values of u_1 , u_2 , ω_1 , ω_2 , and may be treated formally as an identity in u, $\bar{u}$, ω, $\bar{\omega}$. A similar identity arises from g_3 , and so we have

$$F^{(3)}_{pqrs}(u_3 , \omega_3) = \bar{F}^{(3)}_{qpsr}(u_3 , \omega_3),\ G^{(3)}_{pqrs}(u_3 , \omega_3) = \bar{G}^{(3)}_{qpsr}(u_3 , \omega_3),$$
$$(p, q, r, s = 0, 1, 2, \cdots). \tag{3.5}$$

We see that $F^{(3)}_{pprr}(u_3 , \omega_3)$ and $G^{(3)}_{pprr}(u_3 , \omega_3)$ are real; this result is important later in the paper. (The summation convention for repeated suffixes does not operate, here or elsewhere.)

We now substitute the series (3.3) in (3.2), and obtain the following result from the first of (3.2):

$$\sum_{pqrs} F_{pqrs}(u_3 , \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s = \sum_{pqrs} \exp\,[i\alpha(1 - p + q - r + s)]F_{pqrs}(u_3 , \omega_3)u^p\bar{u}^q\omega^r\bar{\omega}^s. \tag{3.6}$$

This may be treated formally as an identity in u, $\bar{u}$, ω, $\bar{\omega}$, and we obtain

$$F_{pqrs}(u_3 , \omega_3) \sin \frac{1}{2}\alpha(1 - p + q - r + s) = 0, \qquad (p, q, r, s = 0, 1, 2, \cdots). \tag{3.7}$$

Thus all coefficients $F_{pqrs}(u_3 , \omega_3)$ must vanish except those for which p, q, r, s are such that

$$\sin \frac{1}{2}\alpha(1 - p + q - r + s) = 0. \tag{3.8}$$

Since $\alpha = 2\pi/n$, we see that *the only surviving coefficients* $F_{pqrs}(u_3 , \omega_3)$ *in the case of n-gonal rotational symmetry are those for which* p, q, r, s *satisfy one of the equations*

$$p - q + r - s - 1 = mn \qquad (m = 0, \pm 1, \pm 2, \cdots) \tag{3.9}$$

(We recall that p, q, r, s are of course non-negative integers.) Similarly, *the only surviving coefficients* $G_{pqrs}(u_3 , \omega_3)$ *in the case of n-gonal rotational symmetry are those for which* p, q, r, s *satisfy one of the equations* (3.9).

A similar argument applied to the third and fourth equations of (3.2) shows that *the only surviving coefficients* $F^{(3)}_{pqrs}(u_3 , \omega_3)$ *and* $G^{(3)}_{pqrs}(u_3 , \omega_3)$ *in the case of n-gonal rotational symmetry are those for which* p, q, r, s *satisfy one of the equations*

$$p - q + r - s = mn \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{3.10}$$

The results are most easily interpreted if we introduce P and Q defined by

$$P = p + r, \qquad Q = q + s. \tag{3.11}$$

We note that as p, q, r, s cover the range $0, 1, 2, \cdots, P$ and Q cover the same range. For a given value of P, there will occur in the power series (3.3) the combinations

$$u^P, \quad u^{P-1}\omega, \quad u^{P-2}\omega^2, \quad \cdots \quad u\omega^{P-1}, \quad \omega^P, \tag{3.12}$$

and for a given value of Q the combinations

$$\bar{u}^Q, \quad \bar{u}^{Q-1}\bar{\omega}, \quad \bar{u}^{Q-2}\bar{\omega}^2, \quad \cdots \quad \bar{u}\bar{\omega}^{Q-1}, \quad \bar{\omega}^Q. \tag{3.13}$$

The result regarding surviving terms may be displayed as follows:

$$\begin{array}{ll} F_{pqrs}(u_3, \omega_3) & F^{(3)}_{pqrs}(u_3, \omega_3) \\ G_{pqrs}(u_3, \omega_3) & G^{(3)}_{pqrs}(u_3, \omega_3) \\ \text{survive only if} & \text{survive only if} \\ P - Q = mn + 1 & P - Q = mn \end{array} \tag{3.14}$$

$$m = 0, \pm 1, \pm 2, \cdots$$

$$(P = p + r, \quad Q = q + s)$$

Since, for small cross velocity and cross spin, the significance of the terms in the series (3.3) decreases with increasing degree, it appears advisable to arrange the terms in order of increasing degree. The degree of a term with coefficient having subscripts p, q, r, s is

$$D = p + q + r + s = P + Q. \tag{3.15}$$

To get a surviving coefficient F_{pqrs} or G_{pqrs} for a term of degree D, we must satisfy the first of the following equations and one of the second:

$$P + Q = D, \quad P - Q = mn + 1 \quad (m = 0, \pm 1, \pm 2, \cdots). \tag{3.16}$$

Equivalently,

$$P = \frac{1}{2}(D + 1 + mn), \quad Q = \frac{1}{2}(D - 1 - mn). \tag{3.17}$$

To get a surviving coefficient $F^{(3)}_{pqrs}$ or $G^{(3)}_{pqrs}$ for a term of degree D, we must satisfy the first of the following equations and one of the second:

$$P + Q = D, \quad P - Q = mn \quad (m = 0, \pm 1, \pm 2, \cdots). \tag{3.18}$$

Equivalently,

$$P = \frac{1}{2}(D + mn), \quad Q = \frac{1}{2}(D - mn). \tag{3.19}$$

The equations (3.16) and (3.18) form a sieve by means of which we reject certain terms from the series (3.3), the existence of the rejected terms being in fact inconsistent

with the assumed symmetry. It is difficult to describe in words the detailed consequences of the operation of this sieve. Some statements will be made below, but the most rapid way to see the results is to consult the diagrams (Figs. 2a–n). These show by heavy dots the surviving terms for symmetries ranging from digonal to octagonal.

To interpret these diagrams, let us take Fig. 2e as an example. This refers to the cross force F and cross torque G in the case of tetragonal symmetry. P and Q are plotted as rectangular Cartesian coordinates; since neither P nor Q can be negative, only the first quadrant is shown. The two families of lines are those given by (3.16) for $D = 0, 1, 2, \cdots$ and $m = 0, \pm 1, \pm 2, \cdots$. The dots are those intersections of these lines which occur at integer or zero values of P and Q. Lines, such as $P + Q = 2$, which give no such intersections, are not shown.

We may analyze the diagram by proceeding through increasing values of the degree D. There is no term of zero degree, nor indeed of any even degree; this is consistent with the general Theorem I, given later.

For terms of the first degree, we have $D = 1$, and an intersection occurs at $P = 1$ and $Q = 0$. Thus, by (3.11), we have surviving terms with

$$p = 1, \quad q = 0, \quad r = 0, \quad s = 0,$$

and

$$p = 0, \quad q = 0, \quad r = 1, \quad s = 0.$$

Consequently there are linear terms in the expansions of F and G of the forms

$$\begin{aligned} &F_{1000}(u_3\ ,\ \omega_3)u + F_{0010}(u_3\ ,\ \omega_3)\omega, \\ &G_{1000}(u_3\ ,\ \omega_3)u + G_{0010}(u_3\ ,\ \omega_3)\omega, \end{aligned} \tag{3.20}$$

respectively, and no other linear terms.

For terms of the third degree, we have $D = 3$, and intersections occur at $(P = 2, Q = 1)$ and $(P = 0, Q = 3)$. Thus we have surviving terms for the following values of p, q, r, s:

p	q	r	s		
2	1	0	0	$P = 2,$	$Q = 1$
1	1	1	0		
0	1	2	0		
2	0	0	1		
1	0	1	1		
0	0	2	1		
0	3	0	0	$P = 0,$	$Q = 3$
0	2	0	1		
0	1	0	2		
0	0	0	3		

Thus the following terms of the third degree occur in the expansion (3.3) of F, with similar terms in the expansion of G:

$$
\begin{aligned}
&F_{2100}(u_3\ ,\ \omega_3)u^2\bar{u} + F_{1110}(u_3\ ,\ \omega_3)u\bar{u}\omega + F_{0120}(u_3\ ,\ \omega_3)\bar{u}\omega^2 \\
&+ F_{2001}(u_3\ ,\ \omega_3)u^2\bar{\omega} + F_{1011}(u_3\ ,\ \omega_3)u\omega\bar{\omega} + F_{0021}(u_3\ ,\ \omega_3)\omega^2\bar{\omega} \\
&+ F_{0300}(u_3\ ,\ \omega_3)\bar{u}^3 + F_{0201}(u_3\ ,\ \omega_3)\bar{u}^2\bar{\omega} + F_{0102}(u_3\ ,\ \omega_3)\bar{u}\bar{\omega}^2 \\
&+ F_{0003}(u_3\ ,\ \omega_3)\bar{\omega}^3.
\end{aligned} \tag{3.21}
$$

There are no other surviving terms of the third degree in these expansions.

In the case of the diagrams for the axial components F_3 , G_3 , the lines shown have the equations (3.18). The reader should have no difficulty in interpreting the other details of the diagrams. Let us now make some general statements with regard to the consequences of n-gonal rotational symmetry.

First, let us consider consequences of evenness or oddness in the order n of the n-gonal rotational symmetry. The following results follow easily from (3.17) and (3.19):

I. *In the case of n-gonal rotational symmetry of EVEN order (digonal, tetragonal, hexagonal, etc.), the series* (3.3) *for F and G contain only terms of ODD degree, and the series* (3.3) *for F_3 and G_3 contain only terms of EVEN degree.*

II. *In the case of n-gonal rotational symmetry of ODD degree (trigonal, pentagonal, etc.), the lowest EVEN degree occurring in the series* (3.3) *for F and G is $(n - 1)$, and the lowest ODD degree occurring in the series* (3.3) *for F_3 and G_3 is n.*

Secondly, let us consider the survival of terms of low degrees ($D = 0, 1, 2$).

If $D = 0$, then $P = Q = 0$. In the case of F and G, we have to satisfy the second of (3.16) in the form $0 = mn + 1$. If $n \neq 1$, this cannot be satisfied by any m in the range $0, \pm 1, \pm 2, \cdots$. If $n = 1$, it is satisfied by $m = -1$. Thus we have the result:

III. *The absolute terms in the series* (3.3) *for F and G survive only in the degenerate case $n = 1$, which corresponds to no real symmetry at all.*

In the case of F_3 and G_3 we have to satisfy the second of (3.18) in the form $0 = mn$. This can always be done by taking $m = 0$. Hence we have the result:

IV. *The series* (3.3) *for F_3 and G_3 always have absolute terms.*

Now let us consider terms of the first degree, $D = 1$. Then either

$$P = 1, \qquad Q = 0, \tag{3.22}$$

or

$$P = 0, \qquad Q = 1. \tag{3.23}$$

In the case of F and G, we have to satisfy the second of (3.16). Corresponding to (3.22) and (3.23) respectively, this gives

$$mn = 0, \tag{3.24}$$

and

$$mn = -2. \tag{3.25}$$

Equation (3.24) is satisfied for any n by taking $m = 0$. Equation (3.25) can be satisfied only if $n = 1$ or $n = 2$, in which cases it has solutions $m = -2$, $m = -1$, respectively. Noting that (3.22) gives

$$p = 1, \qquad q = 0, \qquad r = 0, \qquad s = 0,$$

$$p = 0, \qquad q = 0, \qquad r = 1, \qquad s = 0,$$

and (3.23) gives

$$p = 0, \quad q = 1, \quad r = 0, \quad s = 0,$$
$$p = 0, \quad q = 0, \quad r = 0, \quad s = 1,$$

we may state the following result:

V. *For any n, the series* (3.3) *for F and G contain linear terms of the form*

$$F_{1000}(u_3 , \omega_3)u + F_{0010}(u_3 , \omega_3)\omega,$$
$$G_{1000}(u_3 , \omega_3)u + G_{0010}(u_3 , \omega_3)\omega. \tag{3.26}$$

If $n > 2$, *these are the only terms of the first degree; if* $n = 1$ *or* $n = 2$, *there are additional linear terms of the form*

$$F_{0100}(u_3 , \omega_3)\bar{u} + F_{0001}(u_3 , \omega_3)\bar{\omega},$$
$$G_{0100}(u_3 , \omega_3)\bar{u} + G_{0001}(u_3 , \omega_3)\bar{\omega}, \tag{3.27}$$

respectively.

To discuss terms of the first degree in the series for F_3 and G_3, we go back to (3.22) and (3.23), and combine them with the second of (3.18). We get respectively,

$$mn = 1, \tag{3.28}$$

or

$$mn = -1. \tag{3.29}$$

These can be satisfied if and only if $n = 1$. Thus we have the result:

VI. *In the series* (3.3) *for* F_3 *and* G_3, *linear terms occur only in the degenerate case* $n = 1$.

Finally let us consider terms of the second degree, $D = 2$. Now we have three alternatives:

$$P = 2, \quad Q = 0, \tag{3.30}$$

or

$$P = 1, \quad Q = 1, \tag{3.31}$$

or

$$P = 0, \quad Q = 2. \tag{3.32}$$

In the case of F and G we have correspondingly from the second of (3.16):

$$mn = 1, \tag{3.33}$$

or

$$mn = -1, \tag{3.34}$$

or

$$mn = -3. \tag{3.35}$$

The first two can be satisfied if and only if $n = 1$. Equation (3.35) can be satisfied if and only if $n = 1$ or $n = 3$. Thus we have the result:

VII. *The series* (3.3) *for F and G contain terms of the second degree if and only if* $n = 1$ *or* $n = 3$ (*no symmetry or trigonal symmetry*). *If* $n \neq 1$, *all terms of the second degree involve* $\bar{u}^2$, $\bar{u}\bar{\omega}$, $\bar{\omega}^2$ *only.*

To discuss terms of the second degree in the series for F_3 and G_3, we use (3.30), (3.31), (3.32) with the second of (3.18). We get the following corresponding equations:

$$mn = 2, \tag{3.36}$$

or

$$mn = 0, \tag{3.37}$$

or

$$mn = -2. \tag{3.38}$$

Equation (3.37) can be satisfied for any n; the other two equations can be satisfied if and only if $n = 1$ or $n = 2$. Hence we have this result:

VIII. *If $n > 2$, the series* (3.3) *for F_3 and G_3 possess terms of the second degree of the following forms only*:

$$\begin{aligned} &F^{(3)}_{1100}(u_3, \omega_3)h\bar{u} + F^{(3)}_{1001}(u_3, \omega_3)u\bar{\omega} + F^{(3)}_{0110}(u_3, \omega_3)\bar{u}\omega + F^{(3)}_{0011}(u_3, \omega_3)\omega\bar{\omega}, \\ &G^{(3)}_{1100}(u_3, \omega_3)h\bar{u} + G^{(3)}_{1001}(u_3, \omega_3)u\bar{\omega} + G^{(3)}_{0110}(u_3, \omega_3)\bar{u}\omega + G^{(3)}_{0011}(u_3, \omega_3)\omega\bar{\omega}. \end{aligned} \tag{3.39}$$

If $n = 1$ or $n = 2$ (no symmetry or digonal), all terms of the second degree occur.

The following result may be added as an immediate deduction from the second of (3.16):

IX. *Except in the degenerate case ($n = 1$), there occur in the series* (3.3) *for F and G no terms making*

$$p + r = q + s.$$

This means that, in the diagrams of Fig. 2 which refer to F and G, we find no dots on the main diagonal, $P = Q$.

4. Consequences of reflectional symmetry. Let us now suppose that the projectile has reflectional symmetry in a plane P. If there is also n-gonal rotational symmetry about an axis A, we shall suppose that A is contained in P.

Let us choose the origin O in P (and on A if there is n-gonal rotational symmetry). Let us choose the axis Ox_3 in P (and along A if there is n-gonal rotational symmetry). Finally, let us choose the axis Ox_1 in P, so that Ox_2 is perpendicular to P.

The covering operation is a reflection in P. Applied to the axes, this operation changes $Ox_1x_2x_3$ into $Ox_1'x_2'x_3'$, where Ox_1', Ox_3' coincide with Ox_1, Ox_3 respectively, and Ox_2' is Ox_2 reversed.

Consider any state of motion of the projectile. This may be described by the components $(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3)$ along $Ox_1x_2x_3$ or by the components $(u_1', u_2', u_3', \omega_1', \omega_2', \omega_3')$ along $Ox_1'x_2'x_3'$. It is understood that a component of angular velocity in a given direction is positive if it corresponds to a positive rotation about that direction. Although we may be accustomed to defining a positive rotation as a right-handed rotation, that definition must not be used here, because we are considering two sets of axes with different orientations. We accept the definition that a rotation is positive if it corresponds to a cyclical rotation of the axes. Thus ω_3 is positive if it tends to turn Ox_1 into Ox_2, and ω_3' is positive if it tends to turn Ox_1' into Ox_2'. It is then easy to see that when we reflect the axes in accordance with the covering operation described above, and describe one single state of motion by components along the two sets of axes, the two sets of components are related by the following formulae of transformation:

$$\begin{aligned} u_1' &= \quad u_1 , \qquad u_2' = -u_2 , \qquad u_3' = \quad u_3 , \\ \omega_1' &= -\omega_1 , \qquad \omega_2' = \quad \omega_2 , \qquad \omega_3' = -\omega_3 . \end{aligned} \tag{4.1}$$

To this state of motion there corresponds an aerodynamic force system, which may be described by the components $(F_1 , F_2 , F_3 , G_1 , G_2 , G_3)$ along $Ox_1x_2x_3$ or by the components $(F_1' , F_2' , F_3' , G_1' , G_2' , G_3')$ along $Ox_1'x_2'x_3'$. As in the case of angular velocities, a component of torque is positive if it tends to produce a cyclical rotation of the axes. Thus G_3 is positive if it tends to turn Ox_1 into Ox_2, and G_3' is positive if it tends to turn Ox_1' into Ox_2'. We have then the following formulae of transformation analogous to (4.1):

$$\begin{aligned} F_1' &= \quad F_1 , \qquad F_2' = -F_2 , \qquad F_3' = \quad F_3 , \\ G_1' &= -G_1 , \qquad G_2' = \quad G_2 , \qquad G_3' = -G_3 . \end{aligned} \tag{4.2}$$

The formulae (4.1) and (4.2) are most easily remembered by the following rule: When the axes are reflected, polar vectors (velocity and force) undergo ordinary reflection, but axial vectors (angular velocity and torque) undergo ordinary reflection followed by a reversal of sense.

In the complex notation of (2.2), the transformations (4.1) and (4.2) read

$$\begin{aligned} u' &= \bar{u}, \qquad u_3' = u_3 , \qquad \omega' = -\bar{\omega}, \qquad \omega_3' = -\omega_3 , \\ F' &= \bar{F}, \qquad F_3' = F_3 , \qquad G' = -\bar{G}, \qquad G_3' = -G_3 . \end{aligned} \tag{4.3}$$

The above formulae refer to the effects of reflection of axes; the idea that the projectile is symmetric is not involved. This idea we now introduce, and substitute from (4.3) in (2.4). We see that reflectional symmetry in the plane Ox_1x_3 implies the following identities:

$$\begin{aligned} f(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= \quad \bar{f}(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3), \\ g(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= -\bar{g}(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3), \\ f_3(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= \quad \bar{f}_3(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3), \\ g_3(\bar{u}, u, u_3 , -\bar{\omega}, -\omega, -\omega_3) &= -\bar{g}_3(u, \bar{u}, u_3 , \omega, \bar{\omega}, \omega_3). \end{aligned} \tag{4.4}$$

Here a bar over a functional symbol implies that the sign of i is changed throughout, i.e. both in the arguments and in any complex coefficients which occur in the function. Since f_3 and g_3 are real, it is of course a matter of indifference whether we use bars for them or not.

We now assume that the functions f, g, f_3, g_3 admit series expansions as in (3.3). Substitution in (4.4) gives

$$\begin{aligned} \sum_{pqrs} (-1)^{r+s} F_{pqrs}(u_3 , -\omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s &= \quad \sum_{pqrs} \bar{F}_{pqrs}(u_3 , \omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s , \\ \sum_{pqrs} (-1)^{r+s} G_{pqrs}(u_3 , -\omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s &= -\sum_{pqrs} \bar{G}_{pqrs}(u_3 , \omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s , \\ \sum_{pqrs} (-1)^{r+s} F^{(3)}_{pqrs}(u_3 , -\omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s &= \quad \sum_{pqrs} \bar{F}^{(3)}_{pqrs}(u_3 , \omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s , \\ \sum_{pqrs} (-1)^{r+s} G^{(3)}_{pqrs}(u_3 , -\omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s &= -\sum_{pqrs} \bar{G}^{(3)}_{pqrs}(u_3 , \omega_3) \bar{u}^p u^q \bar{\omega}^r \omega^s . \end{aligned} \tag{4.5}$$

Since these may be treated as identities in $u, \bar{u}, u_3, \omega, \bar{\omega}, \omega_3$, we deduce

$$\begin{aligned}
F_{pqrs}(u_3, -\omega_3) &= (-1)^{r+s}\bar{F}_{pqrs}(u_3, \omega_3),\\
G_{pqrs}(u_3, -\omega_3) &= -(-1)^{r+s}\bar{G}_{pqrs}(u_3, \omega_3),\\
F^{(3)}_{pqrs}(u_3, -\omega_3) &= (-1)^{r+s}\bar{F}^{(3)}_{pqrs}(u_3, \omega_3),\\
G^{(3)}_{pqrs}(u_3, -\omega_3) &= -(-1)^{r+s}\bar{G}^{(3)}_{pqrs}(u_3, \omega_3),\\
&(p, q, r, s = 0, 1, 2, \cdots).
\end{aligned} \tag{4.6}$$

We note that the exponent $(r + s)$ is the degree in cross spin of a term in the expansions (3.3).

To show the implications of (4.6), we introduce the symbols E and O to represent real functions of u_3 and ω_3 which are respectively *even* amd *odd* in ω_3. We may then show the results as follows:

			$(r+s)$ even	$(r+s)$ odd
F_{pqrs}	and	$F^{(3)}_{pqrs}$	$E + iO$	$O + iE$
G_{pqrs}	and	$G^{(3)}_{pqrs}$	$O + iE$	$E + iO$

(4.7)

In making the above deductions from the assumption of reflectional symmetry, we have made no use of the existence or non-existence of n-gonal rotational symmetry. In fact, the consequences of the two types of symmetry are essentially different in nature. As we saw in Section 3, n-gonal rotational symmetry eliminates terms in the series (3.3). Reflectional symmetry does not eliminate any terms; it merely gives us information with regard to odd or even functional dependence on axial spin ω_3 in the coefficients in (3.3). But in the special case discussed below in Secs. 6 and 7, reflectional symmetry does enable us to eliminate further terms from the expansion G_3.

5. Case of vanishing cross spin ($\omega = 0$). Let us now consider the case where the projectile has no cross spin. This may be either an instantaneous condition in a general motion, or it may be a permanent condition enforced by constraints on the projectile. Thus, if the projectile is mounted in a wind tunnel in such a way that its axis is fixed in direction, then the cross spin vanishes. The projectile may be completely fixed, or it may be free to rotate about its axis. We note that, for wind tunnel discussions, the free air stream is to be used as frame of reference.

For vanishing cross spin, we have $\omega_1 = \omega_2 = 0$, and so $\omega = \bar{\omega} = 0$. Thus all terms disappear from the series (3.3) except those for which $r = s = 0$. In fact, the series read

$$\begin{aligned}
f(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} F_{pq00}(u_3, \omega_3)u^p\bar{u}^q,\\
g(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} G_{pq00}(u_3, \omega_3)u^p\bar{u}^q,\\
f_3(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} F^{(3)}_{pq00}(u_3, \omega_3)u^p\bar{u}^q,\\
g_3(u, \bar{u}, u_3, 0, 0, \omega_3) &= \sum_{pq} G^{(3)}_{pq00}(u_3, \omega_3)u^p\bar{u}^q.
\end{aligned} \tag{5.1}$$

We shall suppose that the projectile has reflectional symmetry in the plane Ox_1x_3 ang also n-gonal rotational symmetry about the axis Ox_3 .

Since $r + s$ is even, it follows from (4.7) as a deduction from reflectional symmetry

$$F_{pq00} = E + iO, \qquad F^{(3)}_{pq00} = E + iO,$$
$$G_{pq00} = O + iE, \qquad G^{(3)}_{pq00} = O + iE. \tag{5.2}$$

Turning to (3.11), we note that $P = p$ and $Q = q$. Figure 2 shows the coefficients which survive in the case of n-gonal rotational symmetry, but now we may read p instead of P and q instead of Q on the axes of the diagrams. As already noted in Theorem IX, on the main diagonal ($p = q$) there are no surviving coefficients in the series for F and G (except in the degenerate case of no symmetry ($n = 1$), which is not shown in the diagrams). But in the series for F_3 and G_3 there are terms on the main diagonal in all cases. The coefficients are of the forms

$$F^{(3)}_{pp00}(u_3 , \omega_3), \qquad G^{(3)}_{pp00}(u_3 , \omega_3). \tag{5.3}$$

From the reality of F_3 and G_3 , it follows (as seen after (3.5)) that the above coefficients are *real*. When this information is compared with (5.2), the truth of the following statement is obvious:

X. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, the coefficients* (5.3) *are respectively real EVEN and real ODD functions of* ω_3 .

6. Case of vanishing axial spin ($\omega_3 = 0$). We next consider the case where the projectile has no axial spin ($\omega_3 = 0$), but may have cross spin. This may be an instantaneous state of motion for a fin-stabilized projectile in free flight, or a permanent state enforced by constraints to prevent axial spin for a projectile mounted in a wind tunnel.

Since an odd function of ω_3 must vanish with ω_3 , it follows from (4.7) that, when $\omega_3 = 0$, the coefficients take the forms:

			$(r + s)$ even	$(r + s)$ odd
F_{pqrs}	and	$F^{(3)}_{pqrs}$	$k(u_3)$	$ik(u_3)$
G_{pqrs}	and	$G^{(3)}_{pqrs}$	$ik(u_3)$	$k(u_3)$

(6.1)

where k is used to denote real functions of u_3 ; k does not, of course, represent a single function. These results are consequences of reflectional symmetry, which assigns odd or even dependence on ω_3 , and of the condition of vanishing axial spin, which eliminates either the real or the imaginary part for any given coefficient.

Since the coefficients in the expansions of F and G are initially to be regarded as unrestricted complex numbers, the terms which survive n-gonal rotational symmetry also survive after application of the above two assumptions. The character of the restriction placed on the coefficients in the series for F and G due to vanishing axial spin may be summed up in the following statement:

XI. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, and having no axial spin, the coefficients* F_{pqrs} , G_{pqrs} *are respectively pure real and pure imaginary functions of* u_3 *for* $(r + s)$ *an even integer. If* $(r + s)$ *is odd, the situation is reversed.*

Further, it is easily seen that the assumption of vanishing axial spin does not eliminate any terms in the expansion of F_3. From (6.1) it is seen that these coefficients $F^{(3)}_{pqrs}$ are pure real or pure imaginary functions of u_3 depending on whether $(r + s)$ is an even or odd integer.

The fact that a projectile possesses reflectional symmetry and no axial spin, when combined with the result (3.5), *does* eliminate additional terms in the expansion of the axial torque G_3 for the projectile. By (3.5), $G^{(3)}_{pprr}(u_3 , \omega_3)$ are real, and so, in particular, $G^{(3)}_{pprr}(u_3 , 0)$ are real. But, by (6.1), $G^{(3)}_{pprr}(u_3 , 0)$ are pure imaginary. Hence

$$G^{(3)}_{pprr}(u_3 , 0) = 0. \tag{6.2}$$

Thus we have this result:

XII. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, and having no axial spin, the series* (3.3) *for* G_3 *contains no terms for which* $p = q$ and $r = s$.

Let us examine this result, proceeding by increasing degree D.

If $D = 0$, we have the coefficient $G^{(3)}_{0000}(u_3 , 0)$, and it vanishes by (6.2).

If $D = 1$, we have the coefficients

$$G^{(3)}_{1000}(u_3 , 0), \qquad G^{(3)}_{0100}(u_3 , 0), \qquad G^{(3)}_{0010}(u_3 , 0), \qquad G^{(3)}_{0001}(u_3 , 0).$$

Obviously, (6.2) tells us nothing about these coefficients. However, as we saw in Theorem VI, these coefficients occur only in the degenerate case $n = 1$.

If $D = 2$, the subscripts of $G^{(3)}_{pqrs}(u_3 , 0)$ may be written as follows:

$$\begin{array}{c} 2000 \\ 0020 \\ *1100 \\ 1001 \\ 0110 \\ *0011 \\ 0200 \\ 0002. \end{array} \tag{6.3}$$

Of these, those marked with a star vanish by (6.2). We are left with 6 terms. Combining this with Theorem VIII and (6.1), we have the following result:

XIII. *For a projectile possessing reflectional symmetry and n-gonal rotational symmetry, and having no axial spin, the terms in the series* (3.3) *for the axial torque* G_3 *may be described as follows:*

a. *There is no absolute term.*

b. *There are linear terms only if* $n = 1$ *(degenerate case).*

c. *If* $n = 1$ *or* $n = 2$, *there are six terms of the second degree, with subscripts as shown by the unstarred entries in* (6.3); *if* $n > 2$, *there are only two terms of the second degree, which may be written*

$$k(u_3)(u\bar{\omega} + \bar{u}\omega) \tag{6.4}$$

where k is a real function.

The last deduction follows from (3.37), since, by (3.5),

$$G^{(3)}_{1001}(u_3\ ,\ 0) = \bar{G}^{(3)}_{0110}(u_3\ ,\ 0), \tag{6.5}$$

and, by (6.1), $G^{(3)}_{1001}(u_3\ ,\ 0)$ is real.

We note, that for $n > 2$, the expansion of G_3 reads

$$G_3 = k(u_3)(u\bar{\omega} + \bar{u}\omega) + (\text{terms of higher degree}). \tag{6.6}$$

7. Case where cross spin and axial spin both vanish ($\omega = 0$, $\omega_3 = 0$). We shall now bring together the situations discussed in Secs. 5 and 6 by supposing that the axial spin ω_3 and the cross spin ω vanish simultaneously. We discuss here only the axial torque G_3.

As a particular case of (6.2), we have

$$G^{(3)}_{pp00}(u_3\ ,\ 0) = 0 \qquad (p = 0,\ 1,\ \cdots). \tag{7.1}$$

This means that when $\omega = \omega_3 = 0$, for a projectile possessing reflectional and n-gonal rotational symmetry, *we are to delete the main diagonals in the charts of* Fig. 2 *for G_3 ; these coefficients no longer survive.* Under these circumstances, let us seek the surviving coefficients of lowest degree in the series (3.3) for G_3. It is then a question of finding the smallest non-negative integer D to satisfy (3.18) with the additional condition $P \neq Q$, since the main diagonal has been eliminated.

The equations to be solved read

$$\begin{gathered} P + Q = D, \qquad P - Q = mn \qquad (m = 0,\ \pm 1,\ \cdots), \\ P \neq Q. \end{gathered} \tag{7.2}$$

These imply

$$D^2 = (P + Q)^2 = (P - Q)^2 + 4PQ = m^2n^2 + 4PQ. \tag{7.3}$$

The possibility $m = 0$ is excluded, since by (7.2) it would give the forbidden relation $P = Q$. Hence the smallest D is given by taking $m^2 = 1$ and either $P = 0$ or $Q = 0$. This smallest value of D is n, and the solutions giving the smallest D are

$$P = n, \qquad Q = 0, \qquad m = 1; \qquad P = 0, \qquad Q = n, \qquad m = -1. \tag{7.4}$$

Let us summarize as follows:

XIV. *A projectile possesses reflectional symmetry and n-gonal rotational symmetry. The cross spin and the axial spin both vanish. Under these conditions the series* (5.1) *for the axial torque G_3 starts with terms of degree n, and the series is of the form*

$$G_3 = -ik(u_3)(u^n - \bar{u}^n) + \text{terms of degree } n + 2. \tag{7.5}$$

To complete the proof of this statement, it follows from (7.4) that

$$G_3 = G^{(3)}_{n000}(u_3\ ,\ 0)u^n + G^{(3)}_{0n00}(u_3\ ,\ 0)\bar{u}^n + \text{terms of degree } n + 2. \tag{7.6}$$

The coefficients are pure imaginaries by (4.7) and complex conjugates by (3.5), since G_3 is real. Thus, in terms of a real function $k(u_3)$, we may write (7.6) in the form (7.5).

The angle of yaw (δ) is the angle between the velocity of the point O and the axis of the projectile. Thus

$$\tan \delta = |u|/u_3 . \tag{7.7}$$

Let θ denote the angle between the cross velocity vector u and the axis Ox_1, which it will be remembered lies in a plane of reflectional symmetry of the projectile. We shall count θ positive when the direction of u is generated from Ox_1 by a positive rotation (Fig. 3). Then

$$u = |u| e^{i\theta}. \tag{7.8}$$

Combining (7.7) and (7.8), we have

$$u = u_3 \tan \delta \cdot e^{i\theta}. \tag{7.9}$$

Substitution in (7.6) gives

$$G_3 = 2u_3^n k(u_3) \tan^n \delta \sin n\theta + \text{terms of order } \tan^{n+2} \delta. \tag{7.10}$$

To summarize:

XV. *The formula* (7.10) *shows explicitly the dependence of the principal part of the axial torque G_3 on the yaw δ and the angle θ for a projectile possessing reflectional symmetry and n-gonal rotational symmetry, in the case where the projectile has no cross spin or axial spin.*

As was to be expected, G_3 is periodic in θ with period $\alpha = 2\pi/n$, the angle of the covering operation.

The formula (7.10) holds even in the degenerate case $n = 1$, when we have no rotational symmetry, only reflectional symmetry.

8. Stability. An interesting question concerning the generation of axial spin arises in respect of (7.10). Suppose that a projectile possessing reflectional symmetry and n-gonal rotational symmetry has no cross spin or axial spin ($\omega = \omega_3 = 0$). Then G_3 is of the form (7.10), from which we see that $G_3 = 0$ if either (a) the cross velocity lies in the plane of reflectional symmetry $Ox_1x_3(\theta = 0)$, or if (b) there in no yaw ($\delta = 0$).

Let us consider the case in which the projectile is yawing ($\delta \neq 0$) so that the vanishing or non-vanishing of the axial torque depends only on θ. We shall then have $G_3 = 0$ for $\theta = m\pi/n$, where $m = 0, \pm 1, \pm 2, \cdots$. For these values of θ, the cross velocity vector lies in a plane of symmetry. If the projectile has n-gonal rotational symmetry and one plane of reflectional symmetry (containing the axis of rotational symmetry), then it necessarily has n planes of reflectional symmetry. The angles between these planes are π/n.

We must observe that, for a given projectile, the axes $Ox_1x_2x_3$ are not uniquely determined. Choosing Ox_3 along the axis of the projectile in the general sense of flight, and deciding to use right-handed axes, we have still a choice of $2n$ consecutive directions differing by an angle π/n for the Ox_1 axis. For a given motion, a change from one such choice of Ox_1 to the next alters θ by π/n, and changes the sign of $\sin n\theta$ in (7.10). Since G_3 is not altered, such a change reverses the sign of the function $k(u_3)$. We must therefore be careful not to attach importance to the sign of $k(u_3)$ until we are sure what axes we are using.

Suppose then that we have made some definite choice of Ox_1, as in Fig. 3. Consider various directions of the cross velocity u. As we rotate u, we get successive positions of

vanishing axial torque, namely, whenever u crosses a plane of symmetry. These will be alternately stable and unstable, in the sense that if the vector u is displaced slightly from one of these positions, the axial torque consequent on this displacement will tend to decrease or to increase the angular displacement between u and the plane of symmetry originally coincident with u. The criterion for stability is that the generated axial torque should rotate the projectile so as to bring the plane of symmetry from which u was displaced back into coincidence with u. This requires a positive G_3 for a positive increment in θ; in other words *the criterion for stability is*

$$dG_3/d\theta > 0. \tag{8.1}$$

Referring to (7.10), we see that we may make the following statement:

XVI. *If* $k(u_3) > 0$, *the projectile is stable with respect to axial spin when yawed in the planes of symmetry* $\theta = 2m\pi/n$ $(m = 0, 1, 2, \cdots)$, *and unstable when yawed in the planes of symmetry* $\theta = (2m + 1)\pi/n$ $(m = 0, 1, 2, \cdots)$. *If* $k(u_3) < 0$, *the situation is reversed.*

We have no a priori knowledge about the sign of $k(u_3)$. Suppose the projectile is to consist of a body of revolution and fins as shown in Fig. 1, the broken lines being included so that the projectile has reflectional symmetry. Let us for the sake of definiteness take Ox_1 in a fin.

In the degenerate case $(n = 1)$, where there is no symmetry except reflectional symmetry (Fig. 4a), we expect stability when the fin is downstream. We note that u represents the cross velocity of the projectile relative to the air, so that $-u$ represents the cross wind relative to the projectile; we may expect that stability occurs when $\theta = \pi$, as shown by the vector marked S in Fig. 4a. Thus, according to the above criterion for stability, in order that (8.1) holds for $\theta = \pi$ we must have $k(u_3) < 0$ when $n = 1$. Figure 4a also shows the relation between the angle θ and stability. The shaded portion of the plane is a region of stability in the sense that if the cross velocity vector u once enters the shaded half-plane it can never leave and tends to settle on the position S. If u is initially in the unstable region (right half-plane) the projectile will rotate so that u leaves the unstable region by the shorter route. These statements are not mathematical deductions; they represent what, from general experience, we would expect to happen in the case of a projectile with a single fin.

In the case $n = 2$ (Fig. 4b), it would generally be agreed that the projectile would turn so as to set the face of the fins at right angles to the cross wind. That is, taking Ox_1 in one of the fins, the two positions for u given by $\theta = \pi/2, 3\pi/2$ are positions of stability and the positions $\theta = 0, \pi$ are positions of instability. In order that this situation may result from the above criterion of stability, it is necessary that $k(u_3) < 0$ when $n = 2$.

In both the above cases, it is interesting to note that the positions of stability are those for which the cross velocity u is coincident with the bisector of the angle between adjacent fins. The question then arises as to whether this holds true for projectiles having more than two fins. This of course could be answered if there were an a priori method for the determination of the algebraic sign of $k(u_3)$. In the absence of such a method, experiment alone can decide. To explain the point at issue, let us discuss the case of a bomb with three fins $(n = 3)$, as shown in Figs. 4c, d.

Let us choose the axis Ox_1 in one of the fins. Two cases can arise, $k(u_3) > 0$ and $k(u_3) < 0$, if we omit from consideration the particular case $k(u_3) = 0$.

Let us consider first the case $k(u_3) > 0$. From Theorem XVI we infer that the projectile is in a position of stable equilibrium when $\theta = 0, 2\pi/3, 4\pi/3$, i.e. when the cross velocity vector u lies in one of the fins (Fig. 4c). Also we see that there is unstable equilibrium when $\theta = \pi/3, \pi, 5\pi/3$, i.e. when the cross velocity vector u bisects the angle between two fins. The shaded regions in Fig. 4c indicate regions of stability, in

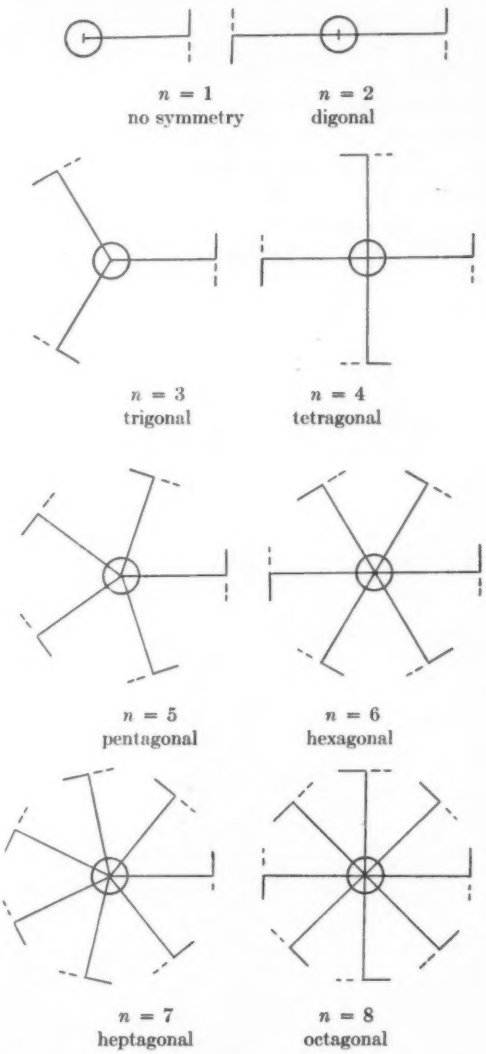

FIG. 1

Diagram showing cross-section with n-gonal rotational symmetry, with and without reflectional symmetry. (If dotted portions are included, we have reflectional symmetry; if they are omitted, we have no reflectional symmetry).

the sense that, if u lies in one of them, the projectile will rotate so as to bring u into coincidence with one of the fins, and there will be an oscillation of u (in the shaded region) about that fin.

On the other hand, if $k(u_3) < 0$, the situation is reversed as shown in Fig. 4d. Now the stable positions of equilibrium are those for which u bisects an angle between fins.

We have then the following interesting question, capable of answer by wind tunnel experiments: If a three finned projectile is mounted in a wind tunnel so that it is free to turn around its axis, are the stable positions of equilibrium those in which a fin lies in the plane of yaw (Fig. 4c), or are they those in which the plane of yaw bisects the angle between two fins (Fig. 4d)? The same question may of course be asked for the case of four fins, or indeed for the case of any number of fins. It would indeed be interesting to know whether the situation discussed for $n = 1$ and $n = 2$ continues for higher values of n. It may be that for higher values of n the screening effect of the body of the projectile becomes of significance, and no general statements can be made covering all projectiles which have the same value of n.

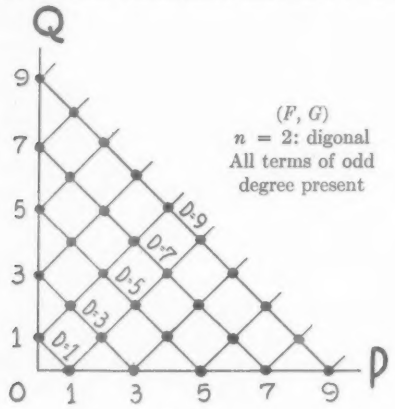

FIG. 2a

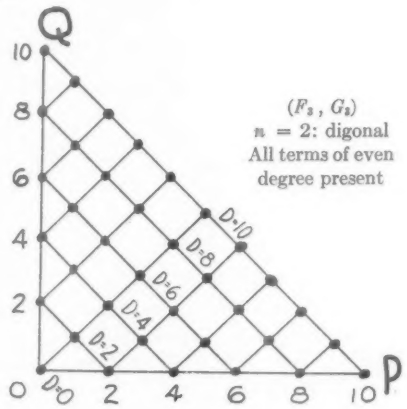

FIG. 2b.

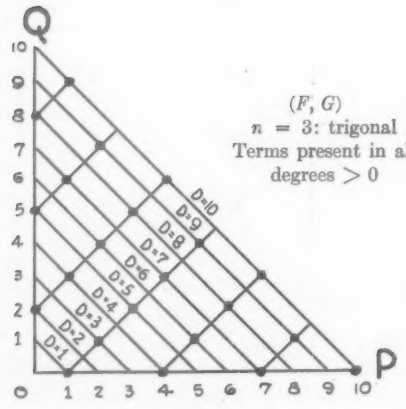

FIG. 2c.

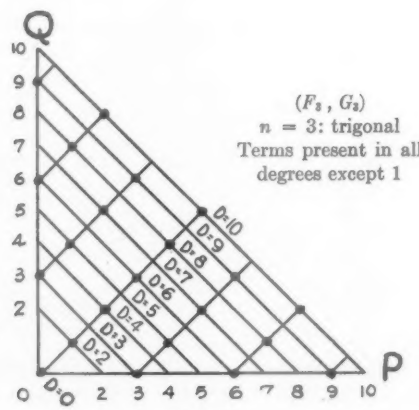

FIG. 2d.

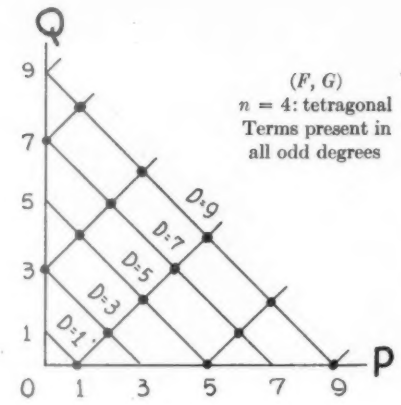

FIG. 2e.

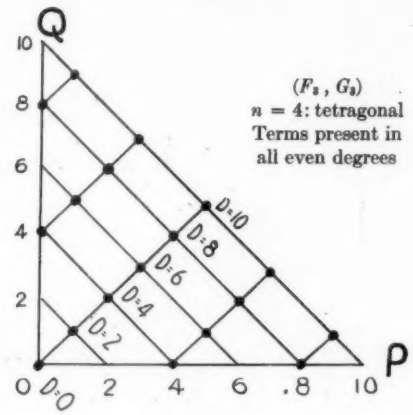

FIG. 2f.

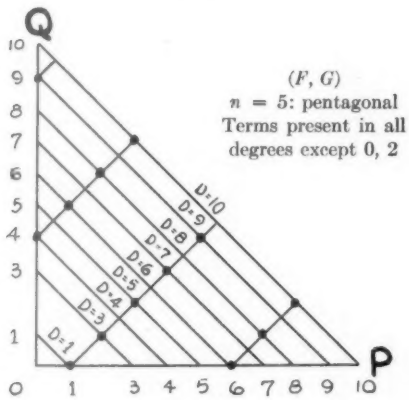

FIG. 2g.

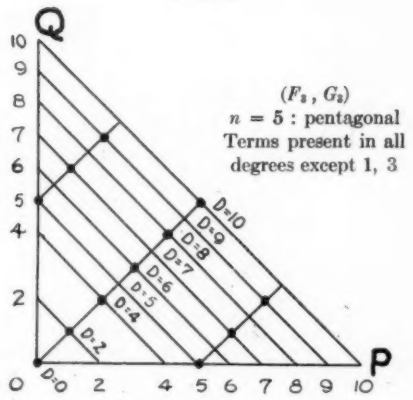

FIG. 2h.

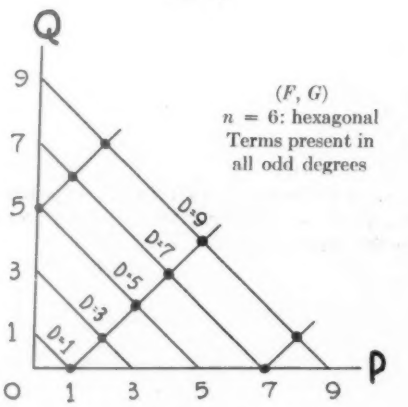

FIG. 2i.

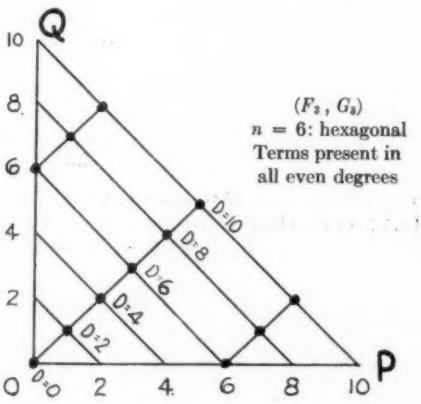

FIG. 2j.

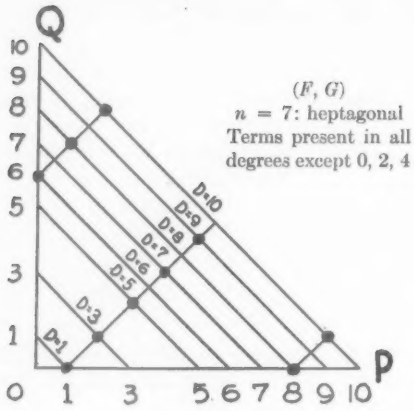

FIG. 2k.

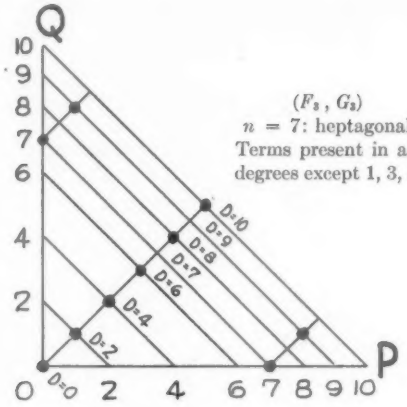

FIG. 2l.

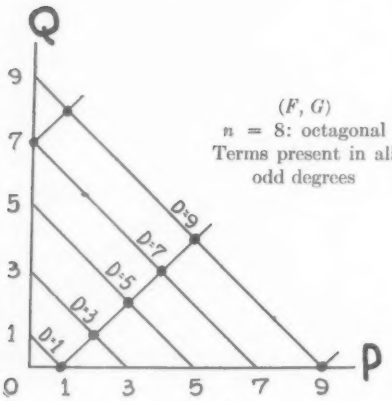

FIG. 2m.

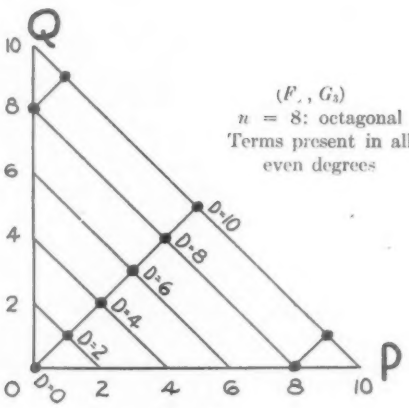

FIG. 2n.

FIG. 2

The heavy dots indicate surviving terms in the expansions (3.3) for various types of rotational symmetry. (F, G) are cross force and cross torque respectively. (F_3, G_3) are axial force and axial torque respectively. For P and Q, see (3.11). D is the degree of the term.

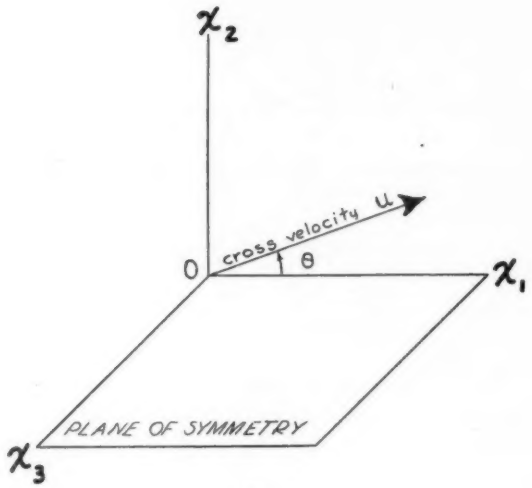

FIG. 3

Projectile with reflectional symmetry; definition of the angle θ.

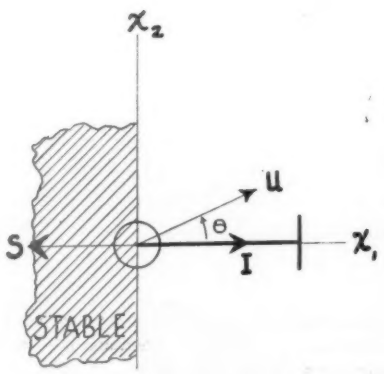

FIG. 4a. $n = 1$, single fin, $k(u_3) < 0$.

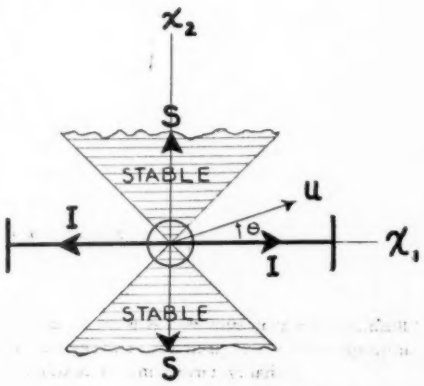

FIG. 4b. $n = 2$, two fins, $k(u_3) < 0$.

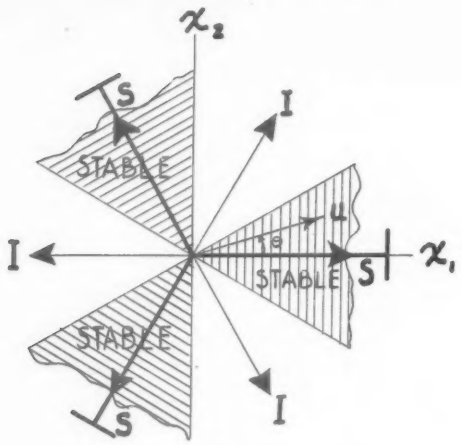

FIG. 4c. $n = 3$, three fins, $k(u_3) > 0$.

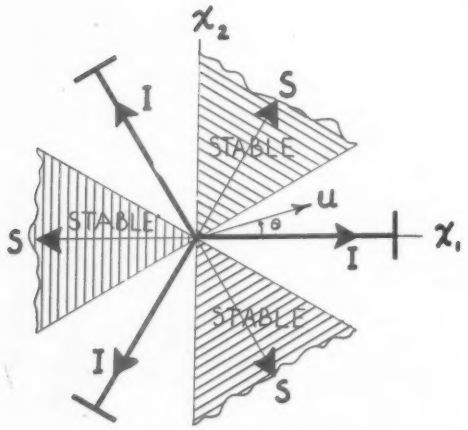

FIG. 4d. $n = 3$, three fins, $k(u_3) < 0$.

FIG. 4

Stability of a projectile with respect to axial spin. S indicates equilibrium directions of u (cross velocity of projectile) which probably correspond to stability. I indicates equilibrium directions of u which probably correspond to instability. Shaded regions are regions of stability.

ON THE STABILITY OF THE SUPERSONIC FLOWS PAST A WEDGE*

BY

G. F. CARRIER

Brown University

1. Introduction. When an ideal[1] compressible gas approaches a symmetrically oriented infinite wedge with uniform supersonic speed, there exist for many wedge angles and upstream velocity magnitudes two alternative steady flow configurations. Associated with one of the flow fields is a plane shock which is attached to the wedge vertex and downstream of which is a uniform subsonic flow. The alternative solution includes a "weaker" attached shock which is more sharply inclined to the wedge surface. Downstream of this shock there also exists a uniform flow field with a larger velocity than that of the strong shock solution. Usually, in this latter case, the downstream flow is supersonic but there is comparatively small range of "inlet" conditions for which it is subsonic. The weak shock is usually observed in experimental work although with sufficient care one can observe the strong shock configuration.[2]

It seems of interest, then, to see what a perfect fluid theory will predict in regard to the stability of these two flows when they are subjected to small time-dependent disturbances.

In the opening sections of this paper we shall be concerned with the interaction of small plane disturbances and plane shock waves. The results of both this plane wave investigation and an independent alternative procedure will then be used to show that both configurations are stable.

2. The wave equations. The fundamental equations which define the motion of an ideal compressible fluid are those implying the conservation of momentum, mass, and energy. A convenient form in which they may be presented is the following [in Eq. (3) we have already used the fact that $p = \rho_0 RT$]:

$$(\mathbf{V} \cdot \text{grad})\ \mathbf{V} + \partial \mathbf{V}/\partial t + \rho_0^{-1} \text{ grad } p = 0, \tag{1}$$

$$\mathbf{V} \cdot \text{grad } (\ln \rho_0) + \partial(\ln \rho_0)/\partial t + \text{div } \mathbf{V} = 0, \tag{2}$$

$$(\mathbf{V} \cdot \text{grad} + \partial/\partial t)(p/\rho_0) = 0. \tag{3}$$

Here V, p, and ρ_0 are respectively, the velocity, pressure, and density, at any point in the field of flow,[3] and γ is the ratio of the specific heats. We shall be interested in the superposition of a small disturbance on a known steady flow. It is therefore convenient to write

$$\mathbf{V} = \mathbf{U}_2 + a_2(\text{grad } \varphi + \text{curl } \mathbf{E}), \tag{4}$$

$$p = p_2(1 + \gamma q), \tag{5}$$

$$\rho_0 = \rho_2(1 + \rho). \tag{6}$$

*Received Sept. 6, 1947; revised March 1, 1948.

[1] We imply vanishing viscosity and heat conductivity, and the validity of the ideal gas law, $p = \rho RT$.

[2] A. Kantrowitz has successfully produced the strong shock solutions.

[3] The subscript zero merely distinguishes the density from the variable without subscript which will soon arise.

Here, the subscript 2 characterized the state parameters associated with the steady flow downstream of the shock. The disturbance is characterized by the functions, φ, q, ρ, and $\mathbf{E}$. No justification is needed for the foregoing representation of the disturbance velocity except to note that *any* vector can be expressed in this manner. In fact, we may (and shall) still require without loss of generality that div $\mathbf{E}$ should vanish identically.

Using now a perturbation process to linearize Eqs. (1), (2), (3), with regard to the disturbance parameters,[4] we obtain the following linear differential equation

$$\Delta\varphi - d^2\varphi/dt^2 = 0, \tag{7}$$

$$d\mathbf{E}/dt = 0, \tag{8}$$

$$dS/dt = 0. \tag{9}$$

It is implied in the manipulations which lead to these equations that

$$q = -d\varphi/dt. \tag{10}$$

In Eqs. (7) to (10) (and henceforth) t represents the time multiplied by a_2 ; M is the Mach number $|\mathbf{U}_2|/a_2$; Δ is the Laplace operator; $d/dt = \partial/\partial t + \mathbf{M}_2 \cdot \text{grad}$; and $S = q - \rho$.

It is of interest to interpret these equations from a physical point of view. Equation (7) implies that the function φ represents an acoustic wave which propagates according to a conventional wave equation. The vector $\mathbf{E}$ characterizes that part of the velocity vector which implies a non-vanishing vorticity and, according to Eq. (8), drifts with the stream. Finally, S, the linearized entropy, also drifts with the stream. The fact that these latter quantities propagate in this manner is, of course, well known [4].

3. The boundary conditions at the shock. In this section we shall be concerned with the formulation of the conditions which must apply at the downstream side of the shock when the steady flow field is disturbed in the immediate neighborhood of the shock. We shall be interested only in those situations where the upstream field remains undisturbed (i.e. either the disturbance originates in the downstream field or sufficient time has elapsed since its formation so that the upstream field is again free from any perturbations). Since, for a stationary shock, the Rankine-Hugoniot conditions[5] uniquely determine the state variables on the downstream side in terms of the upstream state variables, no time dependent changes in the field can occur unless we admit oscillations of the shock locus which have a magnitude consistent with that of the disturbance functions. At any time t, then, the shock is displaced from its steady state position by an amount[6] $\psi = \psi(y, z, t)$. In order to determine the downstream state variables we must apply the Rankine-Hugoniot conditions to a shock moving with a velocity ψ_t and oriented with regard to its equilibrium configuration at angles ψ_y , ψ_z . Here, we have implied the omission of higher order terms[7] in ψ_y , ψ_t , $\cdots$. This linearization is essential if the procedure is to be consistent with that of Sec. 2.

---

[4]Details of such a process may be found in [4]. The most naive approach, however, gives the same result as a rigorous analysis.

[5]See [1], [2], [3].

[6]Here, we choose rectangular coordinate axes as follows: the axis x is normal to the plane of the shock, the plane $x = 0$ is normal to both $\mathbf{U}_1$ and $\mathbf{U}_2$.

[7]The angles, of course, are actually arctan ψ_y , $\cdots$.

Before writing down these jump conditions, the following notation must be introduced.[8]

$$\mathbf{U}_2 = \mathbf{i}u_2 + \mathbf{j}v_2 ; \qquad \mathbf{U}_1 = \mathbf{i}u_1 + \mathbf{j}v_1 ;$$

u_1^*, v_1^*, w_1^*, u_2^*, v_2^*, w_2^*, are the upstream and downstream components of velocity in-

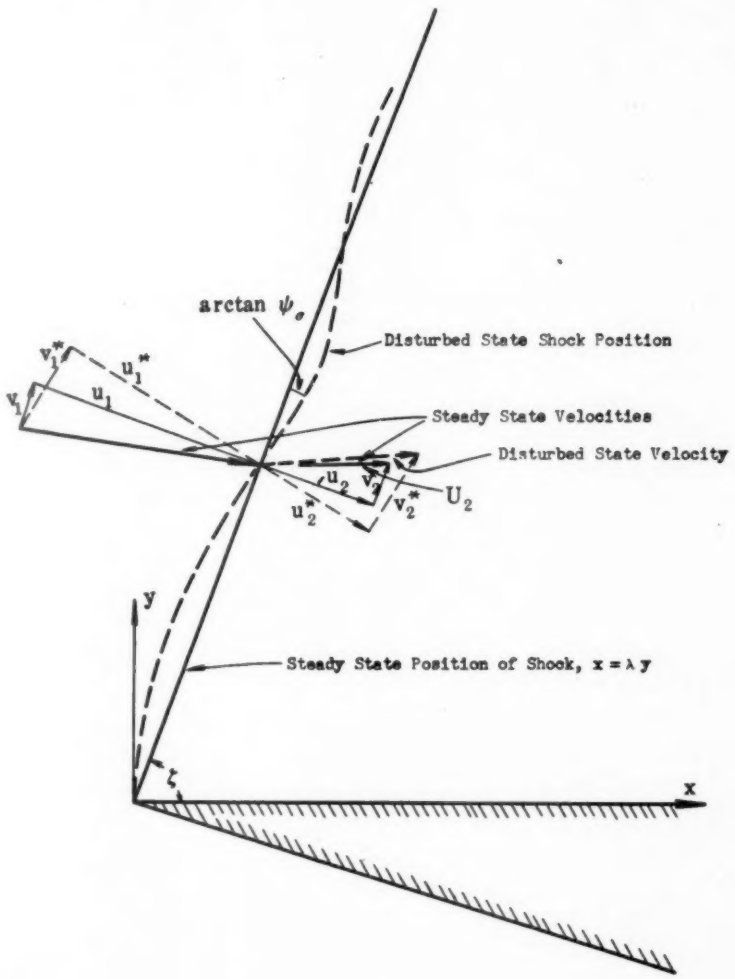

FIG. 1

The coordinate system depicted here is applicable only to the work of Sec. 5. The earlier work utilizes $x = 0$ as the shock locus. In this diagram, of course, the z components of velocity are not visible.

[8] i, j, k will denote the conventional Cartesian unit vectors.

stantaneously normal to and tangent to (in planes $z = 0$ and $y = 0$, respectively) the displaced shock locus; δu_2 , δv_2 , δw_2 , $\delta \rho_2$, δp_2 , are the changes in u_2 , $\cdots$, p_2 , associated with the shock displacement. From Fig. 1, it is readily seen that (after linearization with regard to ψ_y , ψ_z)

$$u_1^* = u_1 - v_1\psi_y , \qquad v_1^* = v_1 + u_1\psi_y , \tag{11}$$

$$u_2^* = u_2 - v_2\psi_y + \delta u_2 , \qquad v_2^* = v_2 + u_2\psi_y + \delta v_2 , \tag{12}$$

$$w_1^* = u_1\psi_z , \qquad w_2^* = u_2\psi_z + \delta w_2 . \tag{13}$$

The Rankine-Hugoniot conditions may now be written in the forms

$$\rho_1(u_1^* - \psi_t) = (\rho_2 + \delta\rho_2)(u_2^* - \psi_t), \tag{14}$$

$$p_1 + \rho_1(u_1^* - \psi_t)^2 = p_2 + (\rho_2 + \delta\rho_2)(u_2^* - \psi_t)^2, \tag{15}$$

$$\gamma R(T_1 - T_2)/(\gamma - 1) + (u_1^* - \psi_t)^2/2 = \frac{\gamma p_2}{(\gamma - 1)\rho_2}(\delta p_2/p_2 - \delta\rho_2/\rho_2) + (u_2^* - \psi_t)^2/2 \tag{16}$$

$$v_1^* = v_2^* , \qquad w_1^* = w_2^* . \tag{17}$$

If we now substitute Eqs. (11) to (13) into Eqs. (14) to (17), linearize with regard to all disturbance parameters (i.e. δu_2 , δp_2 , $\cdots$, ψ_y), and utilize the identities[9] $\rho_1 u_1 = \rho_2 u_2$, $\cdots$, we obtain

$$\frac{\delta u_2}{a_2} = \frac{a_2^2 + u_2^2 - (\gamma - 1)u_1u_2}{a_2^2 - u_2^2}(1 - \rho_1/\rho_2)\left(\psi_t + \frac{v_2}{a_2}\psi_y\right) \tag{18}$$

$$\frac{\delta v_2}{a_2} = (1 - \rho_1/\rho_2)(u_1/a_2)\psi_y \tag{19}$$

$$\frac{\delta w_2}{a_2} = (1 - \rho_1/\rho_2)(u_1/a_2)\psi_z \tag{20}$$

$$\frac{\delta p}{\gamma p_2} = -\frac{u_2}{a_2}\cdot\frac{2a_2^2 + (\gamma - 1)(u_2^2 - u_1u_2)}{a_2^2 - u_2^2}\cdot(1 - \rho_1/\rho_2)\left(\psi_t + \frac{v_2}{a_2}\psi_y\right) \tag{21}$$

$$\frac{\delta p}{\gamma p_2} - \frac{o\rho}{\rho_2} = -(\gamma - 1)\frac{u_1 - u_2}{a_2}(1 - \rho_1/\rho_2)\left(\psi_t + \frac{v_2}{a_2}\psi_y\right) \tag{22}$$

The five quantities on the left sides of these equations must be associated with the values of grad φ + curl $\mathbf{E}$, $-d\varphi/dt$, and S, on the plane $x = 0$. In fact, if we now take $\gamma = 1.40$, and define $m = u_2/a_2$, $n = v_2/a_2$, and use the fact that[10] $u_1u_2 = a^{*2} - (\gamma - 1)v_2^2/(\gamma + 1)$, we can write the boundary conditions in the form

[9] These identities, of course, are the steady state Rankine-Hugoniot Conditions.

[10] Here a^* is the critical velocity.

$$[\text{grad } \varphi + \text{curl } \mathbf{E}]_{x=0} = (1 - \rho_1/\rho_2)[\mathbf{i}\alpha_1(\psi_t + n\psi_y) + \mathbf{j}\alpha_2\psi_y + \mathbf{k}\alpha_2\psi_z], \tag{23}$$

$$[-d\varphi/dt]_{x=0} = (1 - \rho_1/\rho_2)\alpha_3(\psi_t + n\psi_y), \tag{24}$$

$$[S]_{x=0} = (1 - \rho_1/\rho_2)\alpha_4(\psi_t + n\psi_y), \tag{25}$$

where

$$\alpha_1 = (2 + 4m^2)/3(1 - m^2), \tag{26}$$

$$\alpha_2 = (5 + m^2)/6m, \tag{27}$$

$$\alpha_3 = -m(5 + m^2)/3(1 - m^2), \tag{28}$$

$$\alpha_4 = -(1 - m^2)/3m. \tag{29}$$

4. The interaction of a plane acoustic wave and a shock. Let us now consider the hypothetical situation where the semi-infinite space $x > 0$ is unobstructed so that a uniform field exists downstream of the shock located at $x = 0$. Furthermore let us consider solutions of Eqs. (7), (8), (9), subject to the boundary conditions (23), (24), (25), which are of the form[11]

$$\varphi = Af(t - \xi x - \eta y - \zeta z) \tag{30}$$

$$\mathbf{E} = (\mathbf{i}B + \mathbf{j}C + \mathbf{k}D)f(t - \sigma x - \eta y - \zeta z) \tag{31}$$

$$S = Ff(t - \sigma x - \eta y - \zeta z) \tag{32}$$

$$\psi = Gf(t - \eta y - \zeta z). \tag{33}$$

This form implies that the disturbance consists of a plane acoustic wave with wave fronts $\xi x + \eta y + \zeta z = \text{const.}$ and of similar vorticity and entropy waves. In order that these waves obey Eqs. (7), (8), (9), one sees by a simple substitution that

$$\xi^2 + \eta^2 + \zeta^2 - (1 - m\xi - n\eta)^2 = 0 \tag{34}$$

and

$$1 - m\sigma - n\eta = 0. \tag{35}$$

Here, we have used the notation of the foregoing section in writing $d/dt = \partial/\partial t + m\,\partial/\partial x + n\partial/\partial y$. The boundary conditions at $x = 0$ are given by Eqs. (23). (24), (25), and in this case (supplemented by the condition div $\mathbf{E} \equiv 0$). They take the form

$$\begin{bmatrix} -\xi & 0 & \zeta & -\eta & -\alpha_1(1-n) \\ -\eta & -\zeta & 0 & \sigma & \alpha_2 \\ -\zeta & \eta & -\sigma & 0 & \alpha_2 \\ 0 & \sigma & \eta & \zeta & 0 \\ 1-m-n & 0 & 0 & 0 & \alpha_3(1-n\eta) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ G \end{bmatrix} = 0 \tag{36}$$

[11] It will turn out that these solutions are certain eigenfunctions. We shall interpret them later.

and

$$F = \alpha_4(1 - n\eta)G. \tag{37}$$

It is readily shown that this implies $B = 0$, and that Eq. (36) can be reduced, with the aid of Eq. (34) and (35), to the form

$$\begin{vmatrix} \mu & 1 - \mu^2 & \dfrac{4 + 8m^2}{5 + m^2} \\ 0 & 1 & 1 \\ 1 & 0 & -2m \end{vmatrix} = 0, \tag{38}$$

where $\mu = [1 - (\eta^2 + \zeta^2)(1 - m^2)/(1 - n\eta)^2]^{1/2}$. (39)

The solution of Eq. (38) is

$$\mu = -m \pm (1 - m^2)/(5 + m^2)^{1/2}, \tag{40}$$

and implies that

$$\xi = (1 - n\eta)[\pm (5 + m^2)^{-1/2} - 2m/(1 - m^2)]. \tag{41}$$

Before attempting to interpret this class of solutions, we must establish some supplementary facts. Using the relations[12]

$$u_1u_2 = a^{*2} - (\gamma - 1)v_2^2/(\gamma + 1), \tag{42}$$

$$(a_0/a_2)^2 = 1 + (\gamma - 1)(m^2 + n^2)/2, \tag{43}$$

and

$$2a_0^2/(\gamma + 1) < u_1^2 + v_1^2 = u_1^2 + v_2^2 < 2a_0/(\gamma - 1). \tag{44}$$

(the first inequality is implied by the second law of thermodynamics) we may conclude that the range of physically possible flows is given by

$$0 \leq n^2 < \infty, \qquad (\gamma - 1)/2\gamma < m^2 \leq 1. \tag{45}$$

This in turn implies that the bracket in Eq. (41) is always negative for such flows. Finally Eq. (40) implies that $1 - \mu^2$ never vanishes or becomes singular; Eq. (39) then implies that $(1 - n\eta)^2$ never vanishes or becomes singular, and hence $(1 - n\eta)$ is always positive. These conclusions require that ξ be always negative [see Eq. (41)] and we have established that no waves of the form given in Eqs. (30) to (33) can be downstream moving waves. That is to say, a negative ξ implies a wave moving in the negative x direction.

The waves implied by the foregoing analysis can now be interpreted in the following manner. A plane acoustic wave with wave fronts $\xi x + \eta y + \zeta z = t$ impinges on the shock at $x = 0$ and reflects *not* as an acoustic wave but as an "entropy wave" S and a "vorticity wave" **E**. In general, of course, an acoustic wave moving into the shock would reflect as a combination of entropy, vorticity, and acoustic waves, and hence the foregoing solutions are those eigen-solutions for which the amplitude of the acoustic reflection vanishes.

[12]Here, a_0 is the stagnation acoustic velocity of the steady downstream field.

If one wished to consider the general reflection problem, the following procedure would be indicated. Eq. (30) would be replaced by

$$\varphi = A_1 f(t - \xi_1 x - \eta y - \zeta z) + A_2 f(t - \xi_2 x - \eta y - \zeta z),$$

where for given η and ζ the values of ξ_1 and ξ_2 are the two roots of Eq. (34). In this problem the incident wave is represented by $A_1 f(t - \cdots - \zeta z)$ and thus A_1 and the function f are known. The boundary value problem of determining the reflection is treated as before, but now the equation replacing Eq. (36) is non-homogeneous and thus there are not eigenvalues to be found but rather one obtains the coefficients A_2/A_1 , C/A_1 , $\cdots$.

5. The wedge flow stability problem. Let us consider the foregoing field of flow where now, however, we insert a solid boundary along the plane $my - nx = 0$ (i.e. along the direction of the vector $\mathbf{U}_2$). We are interested in the existence within the region above this plane (i.e. the wedge surface) of oscillations which occur without the aid of external excitation. The boundary conditions which now apply are those already formulated for the boundary $x = 0$, the condition that the velocity component normal to the wedge surface vanish at the wedge, and the requirement that the wave be outgoing.

Let us now consider that an acoustic disturbance which vanishes outside of a region R $(y_0 < y < y_1)$ exists in the field. This disturbance may be considered as being composed of plane waves.[13] Some of these waves are moving away from the origin. Within a finite time these waves leave the region R and neither they nor their reflections from shock and wedge can return. Any waves moving toward the origin reflect appropriately from wedge and shock and, within a finite time, reach the origin. Here the nature of the combined reflection is not evident. In fact, if the region R is to be disturbed for more than a finite time, the disturbance introduced at the origin must be self-sustaining once it is initiated. It is evident, then, that we must look for an eigenfunction of our problem in a form that defines the behavior near the origin. In the following, we shall not worry about convergence in the large. The solutions considered will be of a type such that the leading terms of the expansions determine the eigenvalue. This will turn out to be such a value that whether the function is defined over the complete region or not, the solutions desired do not exist. We shall first consider those flows for which the down-stream field is supersonic.

It is convenient at this stage to use a coordinate system such that the x-axis is rotated about the z-axis into the upper wedge surface. Using this rotated coordinate system (see Fig. 1), we may write solutions of Eqs. (7), (8), (9), in the form

$$\varphi = f(x, y) \exp \{ic[t - Mx/(M^2 - 1)]\} \cos \alpha z, \tag{46}$$

$$\mathbf{E} = [\{\mathbf{i}F(y) + \mathbf{j}G(y)\} \sin \alpha z + \mathbf{k}H(y) \cos \alpha z] \exp \left\{ic\left[t - \frac{x}{M} - \frac{\lambda y}{M(M^2 - 1)}\right]\right\} \tag{47}$$

$$S = S(y) \exp \{ic[t - x/M - \lambda y/M(M^2 - 1)]\} \cos \alpha z. \tag{48}$$

Here $f(x, y)$ must satisfy

$$f_{xx} - f_{yy}/(M^2 - 1) + k^2 f = 0 \tag{49}$$

[13] Either a discrete set or a continuum of such waves may be needed here.

where $k^2 = c^2/(M^2 - 1)^2 + \alpha^2/(M^2 - 1)$, and $\mathbf{E}$ and S satisfy Eqs. (8), (9) independent of the dependence on y of F, G, H, $S(y)$. We may also write

$$\psi = g(y) \exp \{ic[t - M\lambda y/(M^2 - 1)]\} \cos \alpha z. \tag{50}$$

Finally, defining $\beta^2 = M^2 - 1$, $r^2 = x^2 - \beta^2 y^2$, $\theta = \text{arc tanh } \beta y/x$, solutions of Eq. (49) may be written in the form,

$$f_\nu(x, y) = J_\nu(kr) \cosh \nu\theta. \tag{51}$$

These functions are such that f_ν vanishes at $y = 0$ and is non-singular at the origin if the admissible values of ν are governed by $Re(\nu) > 0$. This latter condition will be critical in determining the stability of the flow. Using Eqs. (46) to (48) and (50), the boundary conditions Eqs. (18) to (22) and the condition div $\mathbf{E} = 0$ may now be expressed in the following form.

$$f_x(\lambda y, y) - \frac{icM}{M^2 - 1} f(\lambda y, y) + H'(y) - \alpha G(y) = A_1 g(y) + B_1 g'(y), \tag{52}$$

$$f_y(\lambda y, y) + \alpha F(y) + \frac{ic}{M} H(y) = A_2 g(y) + B_2 g'(y), \tag{53}$$

$$-\alpha f(\lambda y, y) - \frac{ic}{M} G(y) - F'(y) = A_3 g(y) + B_3 g'(y), \tag{54}$$

$$\frac{ic}{M^2 - 1} f(\lambda y, y) - M f_x(\lambda y, y) = A_4 g(y) + B_4 g'(y), \tag{55}$$

$$S(y) = A_5 g(y) + B_5 g'(y), \tag{56}$$

$$\frac{-ic}{M} F(y) + G'(y) - \alpha H(y) = 0. \tag{57}$$

The number A_i, B_i, given here can readily be deduced from Eqs. (18) to (22) and are functions of two parameters[14] say m and n.

Let us now consider the function

$$f = \sum_{\mu=0}^{\infty} a_\mu f_{\nu+\mu}(x, y) \tag{58}$$

as defined in Eq. (51). This choice for f and the associated choices of $F(y), \cdots S(y)$ are to satisfy Eqs. (38) to (43). We can anticipate that such functions will be appropriate only for certain eigenvalues ν. Let us assume, for the moment that $\nu \neq 0$. Then we must write

$$F(y) = \sum_{\mu=0}^{\infty} b_\mu J_{\nu+\mu}(k\xi y),$$

$$G(y) = \sum_{\mu=0}^{\infty} c_\mu J_{\nu+\mu}(k\xi y),$$

$$\cdots\cdots\cdots\cdots\cdots,$$

where $\xi = (1 + \lambda^2 - M^2)^{1/2}$.

[14] Actually, in Eqs. (18) to (22), they are given as functions of u_1, u_2, v_2, $a_2 \cdots$, but each of these can be written as $u_1 = u_1(m, n), \cdots$.

If we now substitute these functions into Eqs. (52) to (57) and segregate the equalities in the coefficients of $J_{\nu+\mu}(k\,\xi\,y)$ for each μ, we find that the equality in the coefficients of $J_{\nu-1}(k\xi y)$ constitutes a homogeneous condition for the eigenvalue. The remaining equalities constitute non-homogeneous systems from which the a_j , b_j , $\cdots$, for $j > 0$ may be evaluated.

If one carries out the foregoing process, the characteristic equation for ν can be written (after algebraic reduction)

$$\begin{vmatrix} \beta \sinh(\nu - 1) & \theta_0 & B_2 \\ -M \cosh(\nu - 1) & \theta_0 & B_4 \end{vmatrix} = 0, \tag{59}$$

where $\theta_0 = \operatorname{arc\,tanh}(\beta/\lambda)$. It is also implied that the lowest order non-vanishing coefficient b_n corresponds to $n = 1$ whereas the lowest order coefficient in H is d_0 . Thus the boundary condition at $y = 0$ can be satisfied only if $H(0) = 0$, i.e. only if $Re(\nu) > 0$. It can readily be shown that $B_2 < 0$, $B_4 < 0$, $\beta/\lambda < MB_2/\beta B_4 < \lambda/\beta$, for all wedge flows.[15] This leads to the fact that $Re(\nu) < 0$ for all such flows. It is therefore evident that the type of disturbance defined here does not exist.

Before these considerations can be interpreted as implying the non-existence of eigenfunctions of the type sought (i.e sustained oscillations), one more detail must be supplied. It is quite possible that any such eigenfunction might have a branch point at the origin which was not of the type admitted by the Bessel function expansion.[16] For example, one might expect that, near the origin, the wave function behaves like $r^\nu(\ln r)h(\theta)$. That this is not the case can readily be shown. Without algebraic detail, in fact, the procedure is the following. We consider a solution of the differential equation (7) in the form of an expansion whose leading term is $r^\nu(\ln r)h(\theta)$, and define appropriate functions $\mathbf{E}$ and S consistent with this expansion. Then, as with the Bessel function expansion, the equalities among the coefficients of the leading terms which are implied by the boundary conditions lead to a transcendental equation for the eigenvalue ν. Again, as one can readily verify, $Re(\nu)$ is negative[17] for physically possible wedge flows and thus this type of eigenfunction is not admissible. As a matter of fact, it is not even certain that the expansion considered here leads to a function which is defined throughout the field. If not, this consideration is superfluous but in any event an oscillation of this type cannot exist.

At this point we must extrapolate these results and assume that no other type of branch point will lead to admissible solutions. One can readily obtain the general result that for a branch point of type $r^n g(r)$, where $g(r)$ is singular at the origin, $Re(n)$ will be negative. Therefore, using this extrapolation (or, if the reader prefers, this conjecture), we conclude that the flow configurations associated with a supersonic downstream field are stable. This conclusion is, of course, based on the fact that no self-excited oscillations can exist which vanish identically near the origin (the plane wave results imply this) and that no non-singular solutions obey the boundary conditions in the neighborhood of the origin. This, of course, is in agreement with experimental evidence.

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[15] See appendix for details.

[16] This possibility was first suggested to the author by S. S. Shu.

[17] In fact, the same ν_i are associated with this solution as were found for the preceding expansion.

The subsonic case is more easily disposed of. The definition $\beta^2 = 1 - M^2$, $r^2 = x^2 + \beta^2 y^2$, $\theta = \arctan \beta y/x$, will lead to a set of differential equations and their solutions which are analogous to those found by the corresponding procedure in the supersonic case. Again, a set of Bessel functions multiplied this time by trigonometric functions in θ will form a product series solution. However, in order that the waves be outgoing in the subsonic field, we must now use the Hankel functions $H_\nu^{(2)}$. These are singular for all ν, of course, and again this type of series is not admissible. Therefore, with an extrapolation of the same type we used before, we must conclude that the subsonic field is also stable.

In view of the somewhat unusual nature of our results, it seems worth while to summarize briefly without recourse to the physical picture. In the supersonic (subsonic) case, once we have demanded solutions of Eqs. (7) in the form $f(x, y)e^{ict}$, we have a hyperbolic (elliptic) second order partial differential equation with constant coefficients. The homogeneous boundary conditions require $\varphi_n = 0$ on $\theta = 0$, $L(\varphi, \varphi_n) = 0$ on $\theta = \theta_0$, an outgoing wave, and the overriding condition that φ be non-singular at the origin. Here, L represents a linear combination of the argument functions and their derivatives with regard to r, φ_n is the normal derivative of φ. Actually, the first three conditions given here are sufficient to find a unique set of eigenfunctions in either case. The satisfaction of the "overriding" condition (or the lack thereof) then determines the admissibility of these eigenfunctions and in our case is not satisfied. Thus the tentatively chosen functions are not suitable and hence no sustained oscillations exist.

6. Some general comments. It seems of interest now to speculate on the causes of the fact that the weak shock is usually observed in experimental work. Each of the flow configurations will be altered (as indicated in the fore-going) by any externally generated disturbances. In the supersonic case, such disturbances can get into the field only through the boundary layer and even then must propagate downstream within their Mach cone. On the other hand, disturbances originating at the trailing end of the finite wedge may go upstream in a subsonic stream and the *forced* oscillations of the strong shock may be sufficiently great to "push it over" into the other flow. Essentially the same point of view may be stated in the form: A disturbance in the weak shock field has a very small probability of getting near the origin where the overall behavior is determined. In the strong shock field, large disturbances are continually fed towards the origin, and unless such disturbances can be minimized sufficiently, the transition to the weak shock flow will occur.

7. A remark on a previous investigation. In a recent paper [5], J. I. Levinson has treated this same stability problem also using the small time-dependent-perturbation theory. *He, however, considers a sustained excitation proceeding from the upstream field toward the downstream field.* Although it is true that *random* upstream disturbances might initiate downstream perturbations, the proper investigation must not admit *sustained* excitation but must consider only the eigenfunction perturbations. We must therefore consider the physical argument as presented in that paper as incorrect for the problems which are usually of physical interest. Furthermore, in the boundary conditions at the shock, the author utilizes the fact that the shock is wavy but neglects the fact that it is moving. That is, he omits the terms corresponding to ψ_t in our Eqs. (18) to (22). Hence, the actual calculations cannot even be used to find the response of the downstream field to an upstream excitation. His conclusions, which imply in-

stability of all strong shocks and many weak shocks, are consistently contradicted by experimental evidence.*

Appendix

The inequalities which imply $Re(\nu) < 0$ may be developed as follows

$$B_2 = \frac{5 + m^2}{6M} - \frac{(2 + 4m^2)n^2}{3(1 - m^2)M},$$

$$B_4 = -mn\frac{5 + m^2}{3(1 - m^2)},$$

$$\lambda = n/m,$$

$$\beta^2 = n^2 + m^2 - 1 = M^2 - 1.$$

We shall use the inequalities

$$1 > m^2 > 1/7, \qquad n^2 > 1 - m^2.$$

We have, then,

$$-B_2 = +\frac{(4 + 8m^2)n^2 - (5 + m^2)(1 - m^2)}{6M(1 - m^2)} > \frac{(7m^2 - 1)(1 - m^2)}{6M(1 - m^2)} > 0,$$

$$(MB_2/\lambda B_4) - \frac{\beta}{\lambda} = \frac{\beta}{\lambda}\cdot\frac{(4 + 8m^2)n^2 - (5 + m^2)(1 - m^2) - 2\beta^2m^2(5 + m^2)}{2\beta^2(5 + m^2)m^2},$$

$$> \frac{\beta}{\lambda}\cdot\frac{(7m^2 - 1)(1 - m^2)}{2\beta^2n^2(5 + m^2)} > 0.$$

Hence $MB_2/\lambda B_4 > \beta/\lambda$.
Furthermore

$$\frac{MB_2}{\lambda B_4} - \frac{\lambda}{\beta} = \frac{\lambda}{\beta}\cdot\frac{(4 + 8m^2)n^2 - (5 + m^2)(1 - m^2) - 2n^2(5 + m^2)}{2(5 + m^2)n^2}$$

$$= \frac{\lambda}{\beta}\cdot\frac{-(6n^2 + m^2 + 5)(1 - m^2)}{2n^2(5 + m^2)} < 0$$

and the inequality is completed.

We may now define $(-MB_2) = \sinh\alpha$ and $(-\beta B_4) = \cosh\alpha$, so that Eq. (59) reduces to

$$\sinh[(\nu - 1)\theta_0 + \alpha] = 0$$

or

$$\nu\theta_0 = \theta_0 - \alpha.$$

Note added in proof: Certain discussions have occurred while this paper was in proof which indicate that the analysis of the foregoing problem is not complete. Although no eigenfunctions exist which are exponential in time, we must still investigate how an arbitrary initial disturbance grows with time. This problem has been solved for certain special cases and a continued treatment of this stability question will appear as Part II of this paper.

However, we have shown that $\tanh \alpha > \tanh \theta_0$, so for $\tanh \alpha < 1$, $\alpha > \theta_0$ and ν is negative. For $\tanh \alpha = Q > 1$, we have $\alpha = 1/2 \ln (1 + Q)/(1 - Q)$ and $Re(\alpha) = 1/2 \ln (1 + Q)/(Q - 1) = \text{arc tanh } 1/Q$.

Since we have shown $Q^{-1} > \beta/\lambda$ we again have $Re(\nu) < 0$.

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HOMOGENEOUS HARMONIC FUNCTIONS*

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1. Introduction. We consider, in the following, solutions of the Laplace equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{1.1}$$

which are homogeneous of degree n in the coordinates, that is, such that

$$\varphi(Cx, Cy, Cz) = C^n\varphi(x, y, z) \tag{1.2}$$

for any constant C. The case $n = 0$ is of special interest and is considered at great length. As will be explained in the following paper, with proper change of variables, this case finds an important application in the supersonic conical flows.

The relation (1.2) prescribes the variation of φ along lines through the origin; this relation may therefore be used to reduce the number of independent variables in (1.1), converting it to a differential equation in *two* variables. Transforming (1.1) to spherical coordinates, we shall use (1.2) first to eliminate

$$R = (x^2 + y^2 + z^2)^{1/2}, \tag{1.3}$$

thus converting (1.1) to a proper differential equation over the unit sphere $R = 1$; and then to eliminate z, converting (1.1) into a differential equation in the plane $z = 1$.

For $n = 0$ it is shown that by introducing *isothermal coordinates* along the unit sphere we may transform (1.1) into a two-dimensional Laplace equation. In this case the solutions of (1.1) can therefore be expressed in terms of analytic functions of a single complex variable.

For positive or negative integer n the solutions of (1.1), (1.2) may be expressed similarly in terms of analytic functions of a complex variable by either one of two methods: the first consists in differentiating the solutions for $n = 0$ with respect to x, y, z and using inversion; the second, for positive integer n, utilizes Euler's identity for homogeneous functions of degree n:

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\varphi = n\varphi. \tag{1.4}$$

* The case $n = 1$, also of interest from the point of view of conical flows, is discussed at length, and will be applied in the following paper.

2. Harmonic functions of degree zero. In spherical coordinates the Laplace operator may be written in the form

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial R^2} + \frac{2}{R}\frac{\partial\varphi}{\partial R} + B_2(\varphi); \tag{2.1}$$

where B_2 is the second differential operator of Beltrami along the sphere $R = \text{const.}$

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through the point in question. It will be recalled[1] that for any surface S, the operator B_2 can be defined by means of

$$B_2(\varphi) = \lim_{A \to 0} \int \frac{\partial \varphi}{\partial n} ds/A \tag{2.2}$$

where, as shown on Fig. 1, the integration of the normal derivative is carried out over

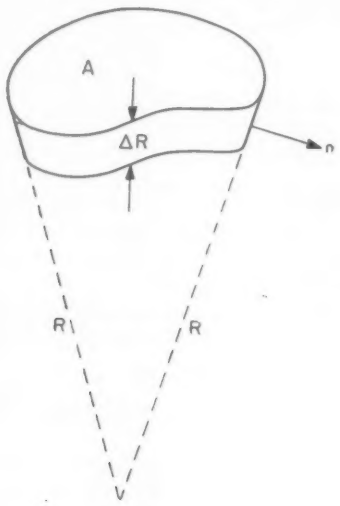

FIG. 1.

the boundary curve C of an area A on S, the normal to the curve being tangent to S, and the limit is taken as the area A shrinks to a point. The form (2.1) is obtained by applying Green's theorem to a volume bounded by two spheres of radii R, $R + \triangle R$ and a conical surface through C, dividing by $\triangle R$ and allowing $\triangle R$ to approach zero.

If φ is homogeneous of degree n, then along the unit sphere $R = 1$, Eq. (2.1) reduces to

$$B_2(\varphi) + n(n + 1)\varphi = 0. \tag{2.3}$$

In particular, for $n = 0$, we are led to

$$B_2(\varphi) = 0 \tag{2.4}$$

over the unit sphere.

To solve (2.4) (along *any* surface S), introduce along S "isothermal" coordinates p, q, that is, coordinates such that

$$ds^2 = \lambda(p, q)[(dp)^2 + (dq)^2]. \tag{2.5}$$

It is known[2] that then the operator B_2 reduces to

$$B_2(\varphi) = (\varphi_{pp} + \varphi_{qq})/\lambda. \tag{2.6}$$

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[1] W. Blaschke, *Vorlesungen ueber Differentialgeometrie*, vol. 1, 2nd ed., J. Springer, Berlin, 1924, pp. 114-116.

[2] *Ibid.*, pp. 121–122.

Thus Eq. (1.1) is reduced to

$$\varphi_{pp} + \varphi_{qq} = 0, \tag{2.7}$$

whose general solution is

$$\varphi = f(p + iq) + g(p - iq) \tag{2.8}$$

or, if φ is real,

$$\varphi = Re\,[f(p + iq)] \tag{2.9}$$

where f, g are general analytic functions of their complex arguments.

It will further be recalled that if one system of isothermal coordinates p, q is available, the general system of such coordinates is given by

$$p' \pm iq' = h(p + iq) \tag{2.10}$$

where h is an arbitrary analytic function of the complex variable $p + iq$; this is in complete agreement with the fact that $B_2 = 0$ reduces to (2.7), and that under the change of variables (2.10), Eq. (2.7) is transformed into a similar Laplace equation in the coordinates p', q'.

Geometrically (2.5) can be interpreted to mean that the surface S is mapped conformally on a plane in which p, q are rectangular Cartesian coordinates. The term "isothermal" is tied up with the fact that Eq. (2.4) is satisfied by the temperature in steady state conduction problems over a thin conducting shell of uniform thickness and thermal conductivity, provided no heat is lost over the faces. In particular, in each isothermal coordinate system, the curves $p = \text{const}$, $q = \text{const}$ may be viewed as a family of isothermals and their normal trajectories.

To render the solution obtained non-vacuous a particular set of isothermal coordinates is required. It will now be shown[3] that a set of isothermal coordinates on the sphere

$$S\colon x^2 + y^2 + z^2 = 1 \tag{2.11}$$

is obtained by projecting a point P on S from the point Q: $(0, 0, -1)$ of S on the negative z-axis onto P' on the equatorial plane $z = 0$; the x, y coordinates of P' in $z = 0$ constitute the parameters p, q of P (see Fig. 2). A projection of P on the tangent plane $z = 1$ at Q': $(0, 0, 1)$ is, of course, also possible, leading to values of p, q which are double those obtained by using the equatorial plane $z = 0$.

Introducing in $z = 0$ the polar coordinates $r = OP' = \tan \psi$ and $\omega = \tan^{-1} y/x$, we have for the element of length $d\sigma$ in that plane

$$\begin{aligned} d\sigma^2 &= dx^2 + dy^2 = dp^2 + dq^2 \\ &= dr^2 + r^2\, d\omega^2 \\ &= \sec^4 \psi\, d\psi^2 + \tan^2 \psi\, d\omega^2. \end{aligned} \tag{2.12}$$

On the sphere S, using the angle $Q'OP = 2\psi = \theta$ and ω as spherical coordinates, we obtain for the element of length

[3] *Ibid.*, p. 168.

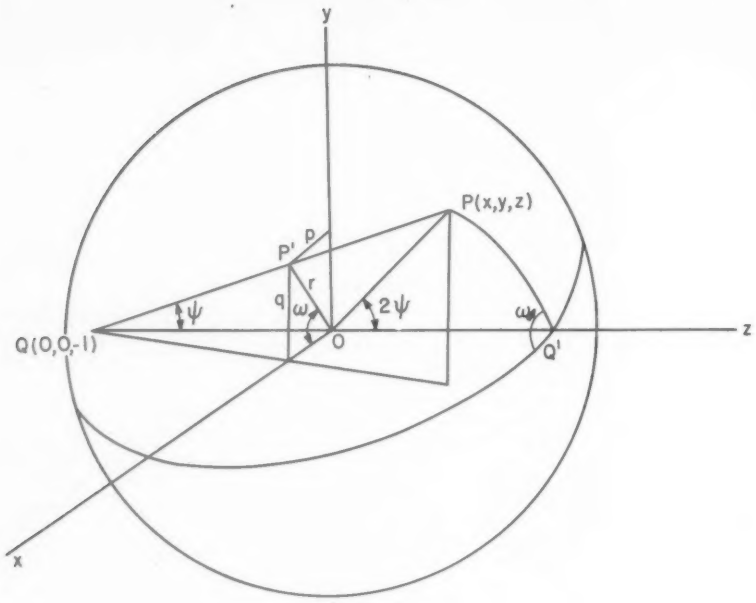

FIG. 2.

$$ds^2 = d(2\psi)^2 + \sin^2 2\psi \, d\omega^2$$

$$= 4(d\psi^2 + \sin^2 \psi \cos^2 \psi \, d\omega^2). \tag{2.13}$$

Comparing (2.13) with (2.12) we obtain

$$ds^2 = 4 \cos^4\psi(dp^2 + dq^2); \tag{2.14}$$

and since

$$\tan^2\psi = r^2 = p^2 + q^2, \tag{2.15}$$

it will be concluded that (2.14) is of the form (2.5), hence that p, q are isothermal coordinates on the unit sphere.

The components of the vector QP on Fig. 2 are x, y, $1 + z$ where x, y, z are the coordinates of P; hence the components of the vector QP' are

$$\frac{x}{1+z}, \qquad \frac{y}{1+z}, \qquad 1. \tag{2.16}$$

Thus the coordinates p, q of P' are given by

$$p = \frac{x}{1+z}, \qquad q = \frac{y}{1+z}, \tag{2.17}$$

while the complex argument of (2.9) becomes

$$+ iq = \frac{x + iy}{1+z}. \tag{2.18}$$

From the above it now follows that the general (real) harmonic function which is homogeneous of degree 0 in x, y, z takes on over the unit sphere the values which form the real (or imaginary) part of

$$f(p + iq) = f\left(\frac{x + iy}{1 + z}\right). \tag{2.19}$$

In order to obtain a representation for this function at arbitrary points of space, we replace x, y, z above by their ratios to R, thus obtaining

$$f\left(\frac{x/R + iy/R}{1 + z/R}\right) = f\left(\frac{x + iy}{R + z}\right) = f(Z) \tag{2.20}$$

where

$$Z = \frac{x + iy}{R + z}. \tag{2.21}$$

It has thus been shown that any real harmonic function which is homogeneous of degree 0 is the real part of an analytic function of Z.

As further examples of isothermal coordinates it is evident that an equally good set of such coordinates on the unit sphere can be obtained by projecting it from *any* one of its points onto the equatorial plane normal to the radius to that point. Thus if we project (2.11) from the point $(0, -1, 0)$ onto the plane $y = 0$ we are led to the complex variable

$$Y = \frac{z + ix}{R + y} \tag{2.22}$$

in place of (2.17), as another possible variable in terms of which the harmonic functions in question may be expressed. The quantity Y must be expressible in terms of the variable Z in accordance with (2.10). Indeed, we may verify that

$$Y = \frac{i - Z}{i + Z}, \qquad Z = (-i)\frac{Y - 1}{Y + 1} \tag{2.23}$$

and this agrees with the fact that the only conformal mapping of a plane on itself (including the "point at infinity" as a "point") is given by a linear fractional transformation. Similarly we obtain from a projection from $(-1, 0, 0)$ on $x = 0$ the variable

$$X = \frac{y + iz}{R + x} = \frac{i - Y}{i + Y} = i\frac{1 - Z}{1 + Z}. \tag{2.24}$$

As a further example consider (see Fig. 2):

$$p' + iq' = \log(p + iq) = \log r + i\omega = \log \tan(\theta/2) + i\omega \tag{2.25}$$

where (θ, φ) are spherical coordinates on S. This isothermal coordinate system consists of small circles (circles of latitude) with Q as pole and meridians through Q. The function

$$\varphi = \log \tan \theta/2 \tag{2.26}$$

corresponds to the temperature solution of a point source at Q' and a point sink at Q. In space the harmonic function (2.26) corresponds to the potential of a uniform positive charge distribution over the positive Z-axis, and a similar negative distribution over the negative z-axis.

The (p, q)-net given by (2.17) consists of circles tangent to the directions of the x or y-axes at Q. The isothermal curves correspond to temperature flow due to a doublet at Q.

3. Alternative derivation of the above results. Let φ be a homogeneous harmonic function of degree zero. Then Euler's identity (1.4) yields

$$x\varphi_x + y\varphi_y + z\varphi_z = 0, \tag{3.1}$$

and when applied to φ_x , φ_y , φ_z which are of degree -1,

$$\begin{aligned} x\varphi_{xx} + y\varphi_{xy} + z\varphi_{xz} &= -\varphi_x , \\ x\varphi_{xy} + y\varphi_{yy} + z\varphi_{yz} &= -\varphi_y , \\ x\varphi_{xz} + y\varphi_{yz} + z\varphi_{zz} &= -\varphi_z . \end{aligned} \tag{3.2}$$

Solving the first two Eqs. (3.2) for φ_{xz} , φ_{yz} and substituting into the first equation, we can express φ_{zz} in terms of x- and y-derivatives of φ as well as in terms of φ_z . Elimination of the latter by means of (3.1) and utilization of

$$x\partial/\partial x + y\partial/\partial y = r\partial/\partial r \tag{3.3}$$

leads to

$$z^2\varphi_{zz} = r^2\varphi_{rr} + 2r\varphi_r ; \tag{3.4}$$

substitution into (1.1) and introduction of cylindrical coordinates r, ω, then yields

$$\varphi_{xx} + \varphi_{yy} + (2r\varphi_r + r^2\varphi_{rr})/z^2 = \left(1 + \frac{r^2}{z^2}\right)\varphi_{rr} + \left(\frac{1}{r} + \frac{2r}{z^2}\right)\varphi_r + \frac{1}{r^2}\varphi_{\omega\omega} = 0. \tag{3.5}$$

For $z = 1$ this reduces to

$$(r^4 + r^2)\varphi_{rr} + (2r^3 + r)\varphi_r + \varphi_{\omega\omega} = 0 \tag{3.6}$$

or in a modified form,

$$\left[\left\{r(1 + r^2)^{1/2}\frac{\partial}{\partial r}\right\}^2 + \frac{\partial^2}{\partial\omega^2}\right]\varphi = 0. \tag{3.7}$$

By introducing the variable s

$$s = \int \frac{dr}{r(1 + r^2)^{1/2}} = \log \frac{r}{1 + (1 + r^2)^{1/2}} \tag{3.8}$$

we may reduce (3.7) to

$$\varphi_{ss} + \varphi_{\omega\omega} = 0, \tag{3.9}$$

leading to the introduction of the analytic functions

$$f(s + i\omega). \tag{3.10}$$

For z other than $z = 1$, Eq. (3.8) is replaced by

$$s = \log \frac{r}{z + (z^2 + R^2)^{1/2}} = \log | Z |, \tag{3.11}$$

leading to

$$s + i\omega = \log Z. \tag{3.12}$$

As a further alternative procedure[4] we may start with spherical coordinates R, θ, ω, writing (2.1) in the form

$$\varphi_{RR} + \frac{2}{R}\varphi_R + \frac{1}{R^2}\left[\left(\frac{1}{\sin\theta}\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\omega^2}\right] = 0, \tag{3.13}$$

and for φ of degree zero, convert to (3.9) by introducing

$$s = \log\tan\theta/2; \tag{3.14}$$

its general solution is exprssible either in the form

$$\varphi = Re[f(se^{i\omega})], \tag{3.15}$$

or in the form (3.10).

4. **Harmonic functions of integer degrees.** The Euler identity (1.4), when applied to a homogeneous harmonic function φ_n of positive integral degree n, yields

$$\varphi = \frac{1}{n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\varphi, \tag{4.1}$$

thus expressing φ in terms of the harnonic functions φ_x , φ_y , φ_z of degree $n - 1$. By successive applications of (4.1) we are led to the identity:

$$\varphi_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)^n\varphi_n , \tag{4.2}$$

where the subscript denotes the degree; this shows that φ_n is a polynomial in x, y, z with coefficients which are harmonic functions of zero degree, so that each one is expressible as the real part of an analytic function, say of Z, as in (2.20). These analytic functions, however, are *not unrelated* to each other. We shall illustrate this for $n = 1$.

Suppose that φ is a real harmonic function of degree 1, then φ_x , φ_y , φ_z are harmonic of degree zero, and hence

$$\varphi_x = Re(U), \qquad \varphi_y = Re(V), \qquad \varphi_z = Re(W) \tag{4.3}$$

where U, V, W are analytic functions of Z. These functions must satisfy the three relations obtained by equating the cross derivatives:

$$\frac{\partial(\varphi_x)}{\partial y} = \frac{\partial(\varphi_y)}{\partial x}, \qquad \frac{\partial(\varphi_y)}{\partial z} = \frac{\partial(\varphi_z)}{\partial y}, \qquad \frac{\partial(\varphi_z)}{\partial x} = \frac{\partial(\varphi_x)}{\partial z}. \tag{4.4}$$

Thus the last equation yields

$$Re\left(\frac{\partial W}{\partial x}\right) = Re\left(\frac{\partial U}{\partial z}\right). \tag{4.5}$$

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[4] H. Bateman, *Partial differential equations of mathematical physics*, Cambridge University Press, 1932, pp. 356-357.

Considering U as a function of Z, we obtain

$$Re\left(\frac{\partial U}{\partial z}\right) = Re\left(\frac{dU}{dZ}\frac{\partial Z}{\partial z}\right); \tag{4.6}$$

now from Eq. (2.21)

$$\frac{\partial Z}{\partial z} = -\frac{x+iy}{(R+z)^2}\left(\frac{z}{R}+1\right) = -\frac{Z}{R}; \tag{4.7}$$

hence there follows

$$Re\left(\frac{\partial U}{\partial z}\right) = -Re\left(\frac{dU}{dZ}\frac{Z}{R}\right). \tag{4.8}$$

Similarly, considering W as a function of X and noting that

$$\frac{\partial X}{\partial x} = -\frac{X}{R}, \qquad \frac{\partial Y}{\partial y} = -\frac{Y}{R} \tag{4.9}$$

we obtain

$$Re\left(\frac{\partial W}{\partial x}\right) = -Re\left(\frac{dW}{dX}\frac{X}{R}\right). \tag{4.10}$$

Upon equating of (4.8), (4.10) and cancellation of R we obtain

$$Re\left(\frac{dW}{dX}X\right) = Re\left(\frac{dU}{dZ}U\right), \tag{4.11}$$

whence

$$\frac{dW}{dX}X = \frac{dU}{dZ}Z + Ci \tag{4.12}$$

where C is a real constant. Dropping the latter, there results

$$U = \int \frac{X}{Z}\frac{dZ}{dX}\,dW. \tag{4.13}$$

Recalling (2.24) it will be noted that

$$\frac{dX}{X} = d(\log X) = -\frac{dZ}{1-Z} - \frac{dZ}{1+Z} = \frac{2dZ}{(Z^2-1)}; \tag{4.14}$$

hence (4.13) becomes

$$U = \frac{1}{2}\int \frac{Z^2-1}{Z}\,dW. \tag{4.15}$$

Similarly (2.23) yields

$$\frac{dY}{Y} = \frac{2idZ}{Z^2+1}, \tag{4.16}$$

and from the second equation (4.4) we now obtain

$$V = \frac{1}{2i}\int \frac{Z^2+1}{Z}\,dW. \tag{4.17}$$

The first relation (4.4) may now be verified from (4.14), (4.17).

Application of (4.1) for $n = 1$ now leads to

$$\begin{aligned}\varphi_1 &= Re\left[\frac{x}{2}\int \frac{Z^2-1}{Z}\,dW + \frac{y}{2i}\int \frac{Z^2+1}{Z}\,dW + zW\right] \\ &= Re\left[\frac{x-iy}{2}\int ZdW - \frac{x+iy}{2}\int \frac{dW}{Z} + zW\right].\end{aligned} \tag{4.18}$$

If the constant C in (4.12) and a similar constant of integration D occurring in the derivation of (4.17) had not been dropped, it would have been necessary to add terms $-C\omega$, $-D\omega$ to u, v. However, the first relation (4.4) shows that $C = D = 0$.

In a similar manner, by utilizing (4.1) for $n = 2$, expressing φ_{xx}, φ_{xy}, φ_{yy} in the form (4.18), then substituting in (4.4), it is possible to express φ_2 in terms of a single analytic function of Z and its proper double quadratures.

Another method of handling homogeneous harmonic functions of integral degree n is by differentiation of functions of degree zero, and spherical inversion. We shall illustrate this first for $n = 1$.

Inversion of (2.20) leads to

$$\varphi_{-1} = Re[f(Z)/R]. \tag{4.19}$$

Upon differentiating (4.19) with respect to z, changing sign, and recalling (4.7), there results

$$\varphi_{-2} = Re\left[\frac{f'(Z)Z}{R^2} + \frac{zf(Z)}{R^3}\right]. \tag{4.20}$$

Further inversion of (4.20) yields

$$\varphi_1 = Re[RZf'(Z) + zf(Z)] \tag{4.21}$$

for real harmonic functions of degree 1.* More generally, differentiating (4.19) n times with respect to z yields

$$\varphi_{-(n+1)} = Re\left\{\frac{\partial^n}{\partial z^n}\left[\frac{f(Z)}{R}\right]\right\} \tag{4.22}$$

for harmonics of negative integral degree, and by inversion of (4.22) we obtain a similar expression for harmonics of positive integral degree.

In particular, choosing

$$f(Z) = 1 \tag{4.23}$$

the above procedure, in a familiar manner, leads to spherical harmonics with axial symmetry involving Legendre polynomials:

$$\varphi_{-(n+1)} = \frac{\partial^n}{\partial z^n}\left(\frac{1}{R}\right) = \text{const. } P_n(\cos\theta)/R^{n+1}. \tag{4.24}$$

*To put (4.21) in a form resembling (4.18) replace R by $z + (x - iy)Z$.

Similarly, by choosing

$$f(Z) = \log Z, \tag{4.25}$$

we obtain axially symmetric spherical harmonics involving Legendre functions of the second kind:

$$\varphi_{-(n+1)} = \frac{\partial}{\partial z^n}\left[\frac{1}{R}\log\frac{x+iy}{R+z}\right] = \text{const. } Q_n(\cos\theta)/R^{n+1}. \tag{4.26}$$

Again, by letting

$$f(Z) = Z^m, \tag{4.27}$$

$$f(Z) = Z^m \log Z \tag{4.28}$$

we are led to harmonics involving the associated Legendre functions.

LINEARIZED COMPRESSIBLE FLOW*

BY

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1. Introduction. The solutions of Laplace's equation indicated in the preceding paper (referred to in the following as I) find ready application in the study of linearized compressible flow. This linearized approximate theory of high-velocity flow is based on the assumption that velocity components induced by an obstacle placed in a uniform stream of high velocity W_0 are small compared to W_0 . Under this assumption the velocity potential φ of the disturbed flow (that is, of the flow induced by the obstacle) can be shown to satisfy the differential equation:

$$\varphi_{xx} + \varphi_{yy} + (1 - M^2)\varphi_{zz} = 0, \tag{1.1}$$

where subscripts denote derivatives, and M is the Mach number of the flow, that is, the velocity of the stream W_0 , assumed in the direction of the z-axis, divided by the velocity of sound at the pressure of the undisturbed flow. The same equation is also satisfied (under the assumption in question) by other thermodynamic gas quantities, for instance by the pressure p which is given by

$$p - p_0 = -W_0\rho_0 w = -W_0\rho_0\varphi_z , \tag{1.2}$$

as well as by the velocity components u, v, w, given by

$$u = \varphi_x , \qquad v = \varphi_y , \qquad w = \varphi_z . \tag{1.3}$$

Equation (1.1) holds both for $M < 1$, that is when the flow is subsonic, as well as for $M > 1$, when the flow is supersonic. In the former case, by introducing coordinates

$$x, y, z = \beta z', \qquad \beta^2 = (1 - M^2) \tag{1.4}$$

one transforms the Eq. (1.1) into the Laplace equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{z'z'} = 0. \tag{1.5}$$

On the other hand, to obtain such a transformation for the supersonic case, it is necessary to carry out the following change of variables:

$$x, y, z = i\beta z', \qquad \beta^2 = M^2 - 1 \tag{1.6}$$

in which real values of z correspond to pure imaginary values of the variable z'. The solutions of I thus have to be re-examined in view of this substitution.

Of much recent interest are supersonic "conical" flows, that is, flows in which the velocity components u, v, w, the pressure p, etc. remain constant along straight lines through a point (the vertex or origin). Such flows are produced by conical obstacles, that is by solids whose boundary consists of straight lines through a vertex. To satisfy the assumptions of the linearized equation (1.1), the conical obstacle must lie close to the z-axis or it may be a thin fin containing the z-axis and inclined at a small angle of attack to the main flow.

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It will be recognized that for supersonic conical flows, u, v, w, $p - p_0$ are homogeneous solutions of Eq. (1.5) of degree zero, while the velocity potential is a solution of Eq. (1.5) of degree 1.

The theory of conical flows is due essentially to A. Busemann,[1] where in the bibliography are given references to earlier papers. Among more recent studies of conical flows are these by H. J. Stewart,[2] R. M. Snow.[3]

While most of the following is devoted to homogeneous solutions of Eq. (1.1) for $M > 1$ of degree zero, solutions of degree n are also discussed. Examples are given both of conical flow and conical flow modified by superposition of other solutions.

The supersonic case $M > 1$ will be understood throughout the following.

2. Conical supersonic flow. To adapt the solutions of I to the solutions of Eq. (1.1) for $M > 1$, in view of Eqs. (1.5), (1.6) the following replacements must be made:

$$z \to z/\beta i, \tag{2.1}$$

$$R = (x^2 + y^2 + z^2)^{1/2} \to (x^2 + y^2 - z^2/\beta^2)^{1/2} = [z^2 - \beta^2(x^2 + y^2)]^{1/2}/\beta i, \tag{2.2}$$

$$Z = \frac{x + iy}{R + z} \to \beta i \frac{(x + iy)}{z + [z^2 - \beta^2(x^2 + y^2)]^{1/2}}. \tag{2.3}$$

From Eq. I. (2.20) one obtains for the general solution of (1.5) the real part of

$$f(Z) = f\left[\frac{\beta i(x + iy)}{z + (z^2 - \beta^2 r^2)^{1/2}}\right]. \tag{2.4}$$

It will also be convenient to introduce the variable

$$\epsilon = \frac{Z}{i} = \frac{\beta(x + iy)}{z + [z^2 - \beta^2(x^2 + y^2)]^{1/2}} = \xi + i\eta = \rho e^{i\omega} \tag{2.5}$$

and this can be used as the argument of f in place of Z; here ξ, η denote the real and imaginary parts of ϵ; ρ its absolute value, ω its argument. Thus, one is led to the solutions

$$f(\epsilon), \qquad \epsilon = \frac{\beta r e^{i\omega}}{z + (z^2 - \beta^2 r^2)^{1/2}}. \tag{2.6}$$

Each velocity component u, v, w, for conical flows can be represented as the real part of an analytic function of ϵ.

Turning to the relations I. (4.4) between the velocity components u, v, w—these relations are equivalent to the vanishing of the curl of the velocity vector—we must keep in mind that while I. (4.15) yields a solution of the equation

$$\frac{\partial U}{\partial z} = \frac{\partial W}{\partial x}, \tag{2.7}$$

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[1] A. Busemann, *Infinitesimale kegelige Ueberschallströmung*, Schriften der deutschen Akademie der Luftfahrtforschung 7B, 105-122 (1943).

[2] H. J. Stewart, *The lift of a delta wing at supersonic speeds*, Q. Appl. Math. 4, 246-254 (1946).

[3] R. M. Snow, *Aerodynamics of thin quadrilatera wings at supersonic speeds*, Q. Appl. Math. 5, 417-428 (1948).

for the *complete* functions U, W (that is, not only for their real parts) in the *old* coordinates, on account of (2.1), it must be replaced by

$$U = \frac{\beta i}{2}\int \frac{Z^2 - 1}{Z}\,dW = -\frac{\beta}{2}\int \frac{\epsilon^2 + 1}{\epsilon}\,dW \tag{2.8}$$

to furnish a solution of (2.7) in the *new* coordinates. Likewise I. (4.17) is replaced by

$$V = \frac{\beta}{2}\int \frac{Z^2 + 1}{Z}\,dW = \frac{\beta i}{2}\int \frac{\epsilon^2 - 1}{\epsilon}\,dW. \tag{2.9}$$

Taking real parts of (2.8), (2.9) to obtain u, v, one obtains

$$u + iv = -\frac{\beta}{2}\left(\int \epsilon dW + \int \frac{d\overline{W}}{\bar{\epsilon}}\right), \tag{2.10}$$

where bars denote conjugates.

For conical flow it is sufficient to determine the velocity components in a plane $z = \text{const.}$, for instance in the plane $z = 1$. For $z = 1$, ϵ reduces to

$$\epsilon = \frac{\beta(x + iy)}{1 + [1 - \beta^2(x^2 + y^2)]^{1/2}} = \frac{\beta r e^{i\omega}}{1 + (1 - \beta^2 r^2)^{1/2}} = \rho e^{i\omega}. \tag{2.11}$$

The radical in Z or ϵ vanishes along the Mach cone

$$r^2 = (x^2 + y^2) = z^2/\beta^2. \tag{2.12}$$

Its section by the plane $z = 1$ yields the "Mach circle"

$$r = 1/\beta, \qquad z = 1 \tag{2.13}$$

and in the ϵ-plane this corresponds to the unit circle

$$\epsilon = e^{i\omega}, \qquad \rho = |\epsilon| = 1. \tag{2.14}$$

Inside the Mach circle, the radicand in (2.11) is positive, and with the positive radical ϵ may be considered to be a map of the Mach circle on the inside of the unit circle $\rho = 1$ with radial direction angle ω preserved, but with a radial distortion.

A geometric representation of this transformation is shown in Fig. 1 where the point P in the plane $z = 1$ is transformed into the point P_2 in the ϵ-plane which has been placed on the $z = 1$ plane. This is done by drawing a sphere of radius $1/\beta$ through the vertex or origin O, with center on the z-axis, projecting P by means of lines parallel to the z-axis on P_1 and P_1' on this sphere, then projecting these from O back on $z = 1$ to P_2 and P_2' respectively. Indeed, note from Fig. 1 that

$$\begin{cases} \sin\alpha = r/(1/\beta) = r\beta, \\ \tan\dfrac{\alpha}{2} = \dfrac{P_2O'}{O'O} = \rho. \end{cases} \tag{2.15}$$

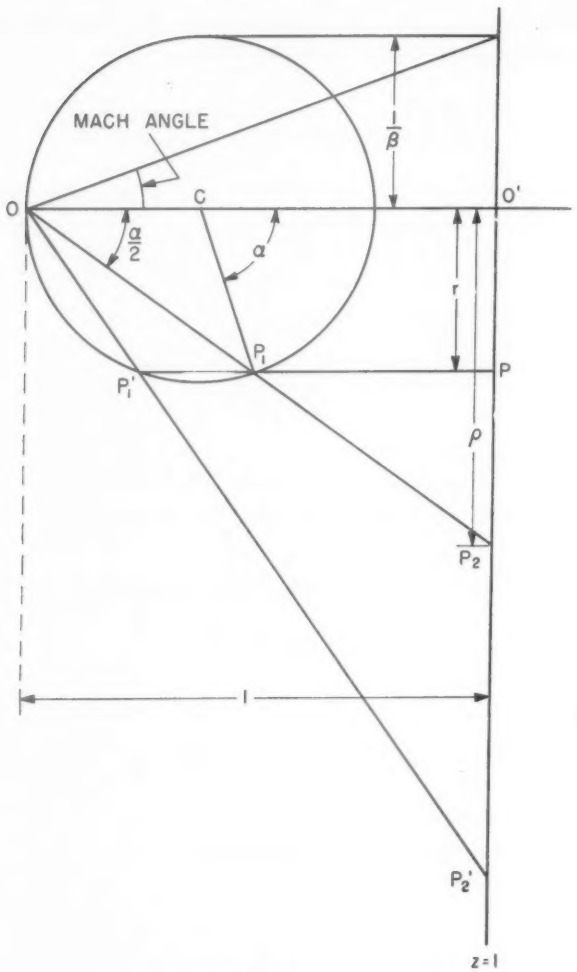

FIG. 1.

Now since

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} \qquad (2.16)$$

$$= \frac{\sin\alpha}{1+\cos\alpha}$$

one obtains from (2.11) and (2.15)

$$\rho = \frac{r\beta}{1 + (1 - r^2\beta^2)^{1/2}} . \tag{2.17}$$

Equation (2.12), as well as the obvious fact that O, P, P_1' , P_2 all lie in the same plane through OO', leads to (2.11).

On the Mach circle, that is for $r\beta = 1$, in the plane $z = 1$, the points P_1 , P_1' coincide, and likewise P_2 , P_2' coincide and lie on the unit circle $\rho = 1$. For P inside the unit circle, two points, P_2 , P_2' are obtained in the ϵ-plane and these correspond to reciprocal values of ρ; these result from (2.17) by using a positive and negative radical.

If P lies *outside* the Mach circle, the construction of Fig. 1 fails; now the radicand in (2.11) is negative and one obtains

$$\epsilon = \frac{\beta r e^{i\omega}}{1 \pm i(\beta^2 r^2 - 1)^{1/2}} . \tag{2.18}$$

As will be noted from Fig. 2, now

$$r\beta = \csc \delta, \tag{2.19}$$

where 2δ is the angle subtended by the Mach circle $r\beta = 1$ at the point P in question, and Eq. (2.18) yields

$$\epsilon = \frac{\csc \delta \, e^{i\omega}}{1 \pm i \cot \delta} = \frac{e^{i\omega}}{e^{\mp i\delta}} = e^{i(\omega \mp \delta)} . \tag{2.20}$$

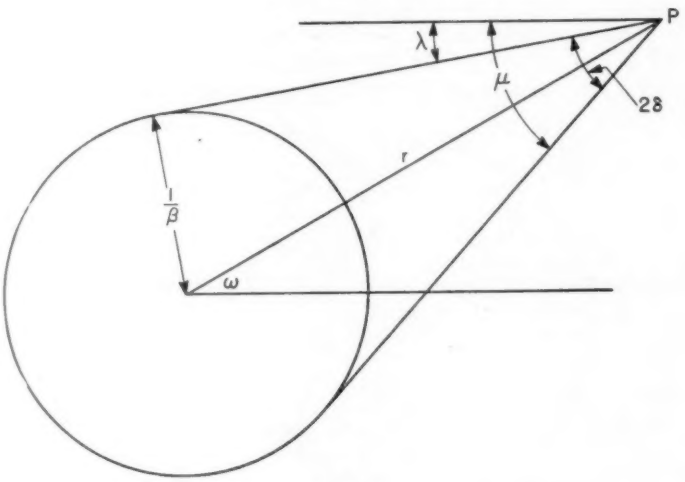

FIG. 2.

Now $\omega - \delta$, $\omega + \delta$ are the angles λ, μ made with the real axis by the tangents from P to the Mach circle. Replacing the argument ϵ in (2.5) by its function $\log \epsilon/i = \lambda, \mu$, we now obtain in place of (2.6)

$$f(\lambda) + g(\mu), \qquad \lambda = \omega - \delta, \qquad \mu = \omega + \delta \tag{2.21}$$

for the velocity components of a conical flow, where f, g are arbitrary real functions; their arguments λ, μ are constant along the tangent lines to the Mach circle.

According to Busemann[1] the introduction of ϵ is due to Chaplygin, whom he also credits with Fig. 1, and Eq. (2.10).

Recalling the alternative procedure of I, Sec. 3, one obtains in a similar fashion for homogeneous zero-degree solutions of Eq. (1.1), or through the substitution of (2.1) in I. (3.5), the following differential equations:

$$\left(1 - \frac{\beta^2 r^2}{z^2}\right)\varphi_{rr} + \left(\frac{1}{r} - \frac{2\beta^2 r}{z^2}\right)\varphi_r + \frac{\varphi_{\omega\omega}}{r^2} = 0 \tag{2.22}$$

or in (x, y)-coordinates

$$\left(1 - \frac{\beta^2 x^2}{z^2}\right)\varphi_{xx} - \frac{2\beta^2 xy}{z^2}\varphi_{xy} + \left(1 - \frac{\beta^2 y^2}{z^2}\right)\varphi_{yy} - \frac{2\beta^2(x\varphi_x + y\varphi_y)}{z^2} = 0. \tag{2.23}$$

It is sufficient to solve these for $z = 1$, and extend the solution to other values of z by replacing r by r/z.

For $z = 1$, Eq. (2.23) is elliptic for $r < 1/\beta$, hyperbolic for $r > 1/\beta$. Its characteristics are given by

$$(1 - x^2\beta^2)dy^2 + 2\beta^2 dx\, dy + (1 - y^2\beta^2)dx^2 = 0, \tag{2.24}$$

whose integration leads to the tangent lines to the circle $r = 1/\beta$ of Fig. 2.

For conical obstacles lying inside the Mach cone, the disturbance field vanishes outside the Mach circle. For conical obstacles outside the Mach cone the disturbance field in the plane $z = 1$ extends to the region enclosed by the characteristics from every point of the obstacle section and the Mach circle itself. Thus for a yawed conical fin shown in Fig. 3, the disturbance in $z = \text{const.}$ covers the horizontally shaded region consisting of the Mach circle and the region between the circle and two of its tangents through the fin end.

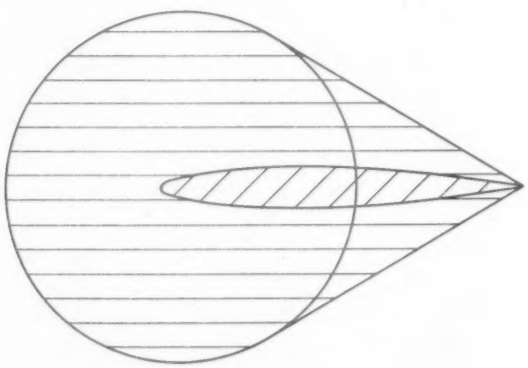

FIG. 3.

By introducing $s = \log \rho$ and ω as variables one converts Eq. (2.22) for $z = 1$ into the Laplace equation as in I. (3.9) for $r\beta < 1$, while for $r\beta > 1$, by λ, μ as in (2.22) one is led to

$$\varphi_{\lambda\mu} = 0. \tag{2.25}$$

3. Flow around cone. The simplest example of a conical flow is obtained by choosing

$$W = A \log \epsilon = A(\log \rho + i\omega) \tag{3.1}$$

where A is a real constant. The real part of this,

$$W = A \log \rho, \tag{3.2}$$

represents an axially symmetric flow: it will be shown that this flow corresponds to the disturbance produced by a circular cone of small angle placed with its axis along the z-axis. Indeed, Eq. (2.10) yields

$$u + iv = -\frac{\beta A}{2}\left(\epsilon - \frac{1}{\epsilon}\right) = \frac{\beta A}{2}\left(\frac{1}{\rho} - \rho\right)e^{i\omega}; \tag{3.3}$$

the factor $e^{i\omega}$ shows that u, v lies in an axial plane and the factor $\rho - 1/\rho$ shows that its magnitude depends on ρ only.

If γ is the (small) cone semi-angle, then the condition

$$\frac{|u + iv|}{W_0} = \tan \gamma = \gamma \tag{3.4}$$

holds along the cone boundary $r = \tan \gamma$ (for $z = 1$) along which Eq. (2.11) yields for small βr

$$\rho = \frac{\beta r}{1 + (1 - \beta^2 r^2)^{1/2}} = \frac{\beta \tan \gamma}{2} = \frac{\beta\gamma}{2}. \tag{3.5}$$

Substituting this value in (3.4), (3.5) and neglecting ρ in comparison with $1/\rho$, one obtains

$$\frac{\beta A 2}{2\beta\gamma} = W_0\gamma, \qquad A = W_0\gamma^2 \tag{3.6}$$

and (3.1) yields

$$W = W_0\gamma^2 \log \epsilon, \qquad w = W_0\gamma^2 \log \rho. \tag{3.7}$$

Substitution from (3.7), (3.8) in (1.2) yields for the pressure rise over the conical surface

$$\frac{p - p_0}{1/2\ \rho_0 W_0^2} = 2\gamma^2 \log \frac{\beta\gamma}{2} = 2\gamma^2 \log\left(\frac{2}{\beta\gamma}\right), \tag{3.8}$$

which essentially agrees with the familiar solutions of this (linearized) problem.

It will be noted from (3.1), (3.3) that along the Mach cone through the vertex the conditions

$$u = v = w = p - p_0 = 0 \text{ for } r\beta = 1,\ p = 1 \tag{3.9}$$

hold. These conditions are satisfied by the conical flow around any conical obstacle lying entirely inside the Mach cone.

The above example is often used to introduce the variables $\log \rho$, θ in a purely geometric and fluid-dynamic manner, namely by means of the pressure field developed around a circular cone.

4. Lift decrease near edge of wing. As a further example, consider the lift decrease near the edges of a wing in a supersonic stream placed (with its span) normally to the main flow. Away from its edges, the well-known two-dimensional flow is set up with a Prandtl-Meyer expansion on the suction side and a pressure shock on the pressure side. The approximate solution for small angles of attack α is obtained from the two-dimensional form of (1.1) (see for instance, Durand[4]) and yields equal and opposite pressure changes $\pm \Delta p$ on both sides, confined to dihedral Mach angles $\cot^{-1}\beta$ to each side of the wing, as shown in Fig. 4, and given by

$$\pm \Delta p = \pm \rho_0 W_0^2 \alpha/\beta. \tag{4.1}$$

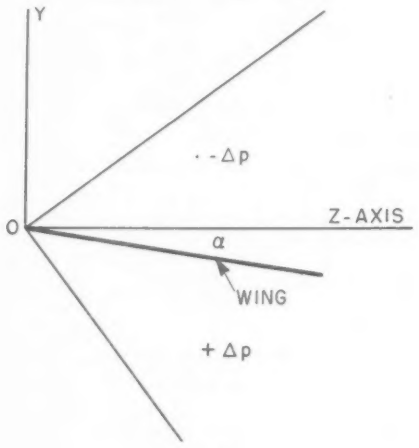

FIG. 4.

Near each edge the flow ceases to be two-dimensional. Placing the forward corner of the wing edge at the origin it is found that the disturbance due to the edge OE is conical. If the wing edge OF lies inside the Mach cone, the disturbance is confined to a Mach cone which is tangent to the dihedral angles.

We shall consider the case of a rectangular wing first (see Busemann[1]). For this case Fig. 5 shows the Mach cones of disturbance at each edge, and Fig. 6 the section by the plane $z = 1$.

Proceeding far enough down stream one comes to the end of the wing and the flow is further disturbed, but this disturbance is only felt down stream and does not affect the conical flow in the Mach cone originating from each forward vertex.

The boundary values for the pressure $p - p_0$ are given in Fig. 6 and are as follows:

$$\frac{p - p_0}{\Delta p} = \begin{cases} -1 \text{ over } AB, \\ +1 \text{ over } AD, \\ 0 \text{ over } BCD. \end{cases} \tag{4.2}$$

Similar conditions apply to w which is related to $p - p_0$ through (1.2). It will be noted

[4]W. F. Durand, *Aerodynamic theory*, Vol. 3, Springer, Berlin, 1935, p. 235.

that (3.9) is violated, which is not surprising since the wing does not lie inside the Mach cone.

In addition to (4.2) there is a condition over AO. To determine the latter note the boundary condition

$$v = -w_0 \tan \alpha = -w_0\alpha, \tag{4.3}$$

which holds to each side of the wing surface, and which we apply over its projection on $y = 0$. Differentiation of (4.3) yields

$$\frac{\partial v}{\partial z} = 0. \tag{4.4}$$

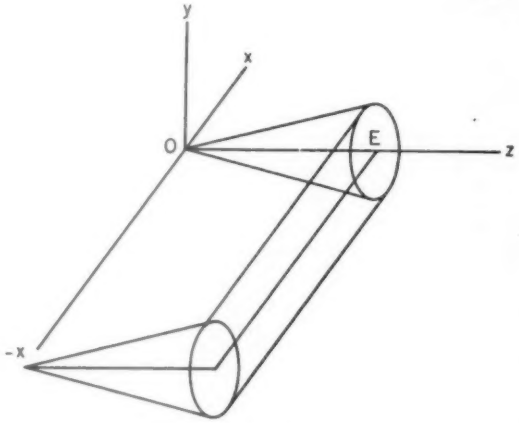

FIG. 5.

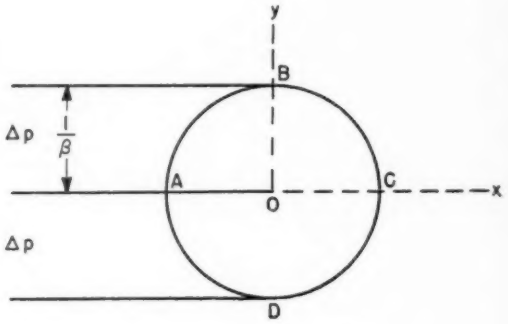

FIG. 6.

From the conditions of vanishing velocity curl, one now obtains from (4.4)

$$\frac{\partial w}{\partial y} = 0. \tag{4.5}$$

This boundary condition enables one to continue w (and hence $p - p_0$ which is proportional to w) from one side of the wing section AO to the other side by means of *positive* reflection across AB. It must be kept in mind that the reflection refers to the analytic continuation of w across the boundary, and not to the flow actually existing on the other side. The condition (4.4) and the reflection is applied separately to the flow above and below AO.

Carrying out a positive reflection for w and p from above AO, and similar reflection from below, it is found that the values obtained for both images are the negatives of the actual values obtaining in the physical flow at the same points. Thus w, p are single-valued over a "two-sheeted Riemann surface" which has a second-order branch point at O. The relation

$$\epsilon_1 = \epsilon^{1/2} \tag{4.6}$$

maps the two-sheeted interior of the unit circle of the ϵ-plane into the single-sheeted interior of the unit circle in the ϵ_1-plane with $OABCDA$ of Fig. 6 going into the semi-circle $OABCDA$ of Fig. 7. In the ϵ_1-plane the boundary values for $(p - p_0)/\Delta p$

are as follows

$$\frac{p - p_0}{\Delta p} = \begin{cases} 0 \text{ over } DCB, \\ -1 \text{ over } BAB', \\ 0 \text{ over } B'D', \\ 1 \text{ over } D'AD. \end{cases} \tag{4.7}$$

Discontinuities for w, p occur at B, $\epsilon_1 = i^{1/2}$ and at D, $\epsilon_1 = -i(i)^{1/2}$ as well as at their images in AOA, B', $\epsilon_1 = i(i)^{1/2}$; D', $\epsilon_1 = -i^{1/2}$. In terms of ϵ_1 we may set up $(p - p_0)$ as follows:

$$\frac{p - p_0}{\Delta p} = \frac{1}{\pi}(\theta_1 - \theta_2 - \theta_3 + \theta_4) = Re\left[-\frac{i}{\pi} \ln \frac{(\epsilon_1 - i^{1/2})(\epsilon_1 + i\{i\}^{1/2})}{(\epsilon_1 - i\{i\}^{1/2})(\epsilon_1 + i^{1/2})} \right] \tag{4.8}$$

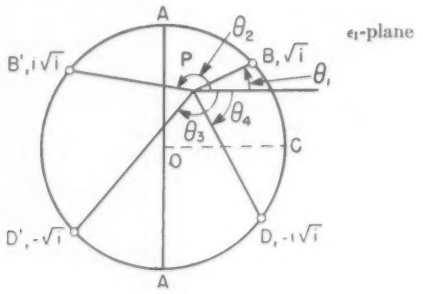

FIG. 7.

where θ_1, θ_2, $\cdots$ as shown in Fig. 9, are the arguments of the factors $\epsilon_1 - i^{1/2}$, $\cdots$ w is obtained by dividing $p - p_0$ by $-W_0\rho_0$. In particular along AA, θ_i simplify; thus

$$\theta_1 = \tan^{-1} \frac{\rho^{1/2} - 2^{-1/2}}{2^{1/2}} = \tan^{-1} [(2\rho)^{1/2} - 1], \tag{4.9}$$

and one obtains from (4.8) for the local lift coefficient along OA of Fig. 5, as a ratio to the lift along the wing proper,

$$\frac{2}{\pi} \tan^{-1} \frac{(2\rho)^{1/2}}{1 - \rho} = \frac{2}{\pi} \sin^{-1} \left[\frac{2\rho}{1 + \rho^2} \right]^{1/2}. \tag{4.10}$$

Utilizing the relation

$$\beta r = \frac{2\rho}{1 + \rho^2} \tag{4.11}$$

which follows from (2.11), one may replace (4.10) by

$$\frac{2}{\pi} \sin^{-1} (\beta r)^{1/2} = \frac{2}{\pi} \cos^{-1} (1 - \beta r)^{1/2} = \frac{1}{\pi} \cos^{-1} (1 - 2\beta r). \tag{4.12}$$

Replacing r by $-x$ and averaging from $x = 0$ to $x = -1/\beta$, one obtains

$$\frac{\beta}{\pi}\int_{-1/\beta}^{0} \cos^{-1}(1 + a\beta x)\,dx = \frac{1}{2\pi}\int_{-1}^{1} \cos^{-1} u\,du = \tfrac{1}{2}. \tag{4.13}$$

The net decrease in lift over the area within the Mach cone due to the edge effect is thus 50%.

We turn now to a trapezoidal wing (see Fig. 8) whose edges make an angle $\pi/2 + \gamma$

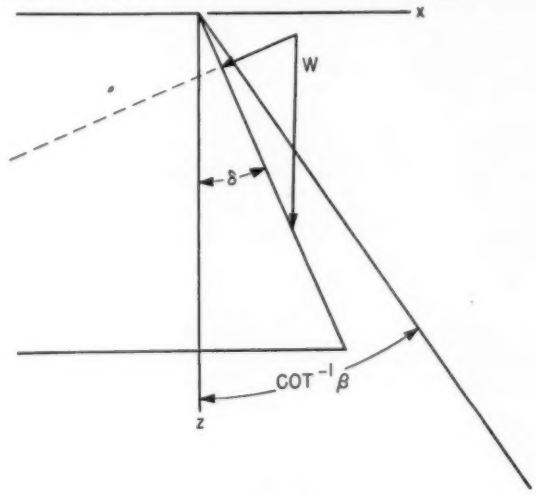

FIG. 8.*

with the forward edge, where

$$0 < \gamma < \cot^{-1}\beta. \tag{4.14}$$

Figure 6 is now replaced by Fig. 9 and in the ϵ-plane the Mach circle is transformed into the unit circle of Fig. 10.

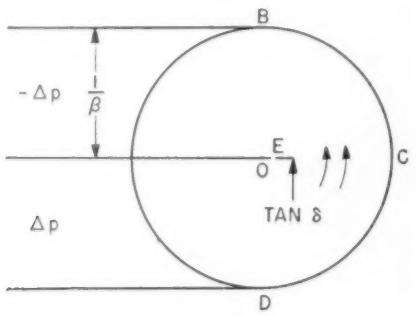

FIG. 9.*

ϵ-plane

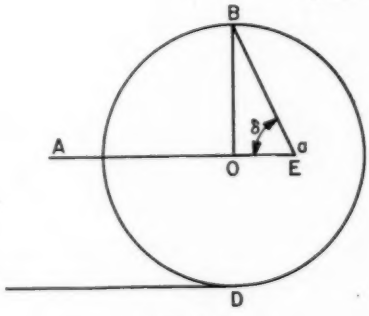

FIG. 10.

with the point E going into $\epsilon = a$ where

*Note added in proof: δ in Figs. 8 and 9 should be replaced by γ.

$$a = \frac{\beta \tan \gamma}{1 - (1 - \beta^2 \tan^2 \gamma)^{1/2}}. \tag{4.15}$$

The boundary conditions along $\rho = 1$ are still given by (4.7), while (4.5) now obtains our AOE. By means of

$$\epsilon' = \frac{a - \epsilon}{a\epsilon - 1} \tag{4.16}$$

one transforms the unit circle in the ϵ-plane into a unit circle in the ϵ'-plane (see Fig. 11)

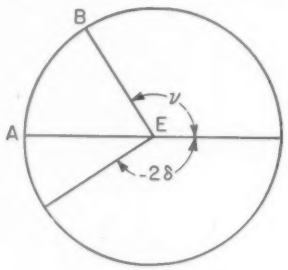

FIG. 11.

with the slit AOE going into the radius along the negative real axis. The point B, $\epsilon = i$ of Fig. 9 goes into

$$\epsilon' = \frac{a - i}{ai - 1} = -i\frac{a - i}{a + i} = e^{i\nu}$$
$$\nu = -2\delta - \pi/2, \qquad \delta = \cot^{-1} a. \tag{4.17}$$

Analytic continuation by means of positive reflection across AE of Figs. 10, 11 is possible and similarly leads to a single valued function in the

$$\epsilon_1 = \epsilon'^{1/2} \tag{4.18}$$

plane, with Fig. 7 replaced by a similar figure but (see Fig. 12) with the discontinuity

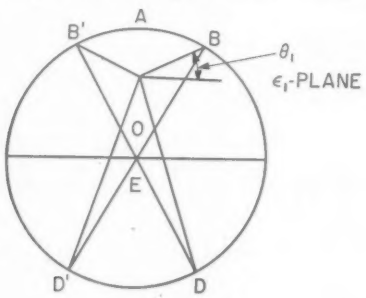

FIG. 12.

points B, B', D, D' replaced by points making angles $\pm\nu/2$ with the directions of positive and negative ϵ_1. A solution analogous to (4.8) is now given by

$$\frac{p - p_0}{\Delta p} = \frac{1}{\pi}(\theta_1 - \theta_2 - \theta_3 + \theta_4) = Re\left[-\frac{i}{\pi}\ln\frac{(\epsilon_1 - e^{i\nu/2})(\epsilon_1 - e^{-i\nu/2})}{(\epsilon_1 + e^{-i\nu/2})(\epsilon_1 + e^{i\nu/2})}\right] \tag{4.19}$$

The solution just obtained (for $p - p_0$ and hence also for w), while it satisfies all the boundary conditions, will not do, however, for the following reason: since the wing possesses lift and builds up a pressure rise and fall $\triangle p$ on the lower and upper sides respectively, there is a tendency to set up circulation around each edge, similar to the circulation of an incompressible fluid around the edges of an airfoil of finite span, and responsible for induced drag. Since the component of the main flow normal to the wing edge, $W_0 \sin \gamma$, is subsonic when (4.14) holds, this flow around the edge (indicated by arrows on Fig. 9) possesses a singularity at the edge similar to that of an incompressible fluid, leading to infinities of the form

$$A(\epsilon - a)^{-1/2} \tag{4.20}$$

in the functions U, V, W. This type of singularity arises also in the case $\gamma = 0$, but there it affects only u and v and not w*. For $\gamma > 0$, the flow lines due to this singular flow lie essentially in planes normal to the edge, and W is also affected by it.

To introduce the required singularity one adds to (4.18)

$$Re\left[\frac{C}{i}\left(\frac{1}{\epsilon_1} - \epsilon_1\right)\right] = C\left(\frac{1}{\rho_1} - \rho_1\right)\sin \omega_1 \,. \tag{4.21}$$

This vanishes for $\rho_1 = 1$ and hence does not interfere with (4.7); it also satisfies (4.5) over AOE of Fig. 9. Thus C in (4.20) cannot be determined from considerations of $p - p_0$, w only. It may be found, however, by means of the relation

$$\left.\frac{dW}{d\epsilon}\right|_{\epsilon=0} = 0 \tag{4.22}$$

without which (4.19), (2.8), and (2.9) would yield singularities for U and W at $\epsilon = 0$. One is lead to

$$C = -\frac{4a}{1 - a}\frac{a - \cos(\nu/2)}{a^2 + 2a\cos\nu + 1} \,. \tag{4.23}$$

This completes the determination of $p - p_0$ and w.

For negative γ with the wing edge still in the Mach cone, a Kutta-Joukowsky condition holds at the edge and the solution (4.19) has to be properly modified. No singularity is introduced at the edge.

The case $\gamma > \cot^{-1}\beta$ has been treated by Snow[3].

5. Non-conical flows. Harmonic functions of various degrees of homogeneity were considered in I, Sec. 4. By means of the substitutions (1.6), (2.1)–(2.3) they can lead to solutions of (1.1) of similar degree of homogeneity.

As an example, familiar harmonic functions of degree n, $-(n + 1)$ of the "product" type are given by

$$[R^n, R^{-(n+1)}]P_n^m(\cos\theta)[\cos m\omega, \sin m\omega], \tag{5.1}$$

where P_n^m are the "associated" Legendre functions, and $\cos\theta = z/R$. Carrying out

*It is introduced by the integrations (2.8)–(2.10).

the indicated substitutions in (5.1), we obtain the following solutions of the wave equation (1.1) for the supersonic case (see Hayes[5]):

$$(z^2 - \beta^2 r^2)^{[n/2,-(n+1)/2]} P_n^m(z/\{z^2 - \beta^2 r^2\}^{1/2})(\cos m\omega, \sin m\omega); \tag{5.2}$$

additional solutions of the same product form may be obtained by utilizing the associated Legendre functions of the "second kind," Q_n^m, in place of P_n^m. In particular, $m = 0$ yields the axially symmetric solutions:

$$(z^2 - \beta^2 r^2)^{[n/2,-(n+1)/2]} P_n[(1 - \beta^2 r^2/z^2)^{-1/2}]. \tag{5.3}$$

The velocity potential for the flow around an axially symmetric body with axis along the z-axis possesses this symmetry.

As a special case for $n = 0$ and the second exponent in the first factor in (5.3), there results

$$(z^2 - \beta^2 r^2)^{-1/2}. \tag{5.4}$$

This is the potential of a "point source" at the origin; it is of degree -1. The velocity *components* for the same symmetric flow, as well as part of the velocity potential due to transverse flow around an axially symmetric body, possess the type of symmetry obtained from (5.3) for $m = 1$.

General solutions of (1.1) of degree n are obtained from I, (4.21), (4.22) by introducing ϵ and the substitutions (2.1) to (2.3).

In particular I.(4.21) yields

$$\varphi = \epsilon(z^2 - \beta^2 r^2)^{1/2} \frac{df(\epsilon)}{d\epsilon} + zf(\epsilon) \tag{5.5}$$

for a general solution of (1.1) of degree 1. The real part of this could be used as a starting point for the velocity potential of conical flows. The requisite boundary condition is given by

$$\varphi_x \lambda_x + \varphi_y \lambda_y + \varphi_z \lambda_z + W_0 \lambda_z = 0, \tag{5.6}$$

where $(\lambda_x , \lambda_y , \lambda_z)$ are the direction cosines of the normal n to the conical boundary. Neglecting $\varphi_z = w$ in comparison with W_0, one may transform the above into

$$\varphi_x \lambda_x' + \varphi_y \lambda_y' = \left(\frac{x}{z}\lambda_x' + \frac{y}{z}\lambda_y'\right) W_0 , \tag{5.7}$$

where $(\lambda_x' , \lambda_y' , 0)$ are the direction cosines of the normal n' to the boundary section by a plane $z = \text{const.}$, this normal lying in that plane (see Fig. 13). Equation (5.7) can be put in the form*

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[5] W. D. Hayes, *Linearized supersonic flows with axial symmetry*, Q. Appl. Math. 4, 255-261 (1946).

*In principle, the singularity of the flow field of Section 8 at A, A' could be obtained by replacing the wing with its sharp edges by a boundary of finite curvature, solving flow subject to (5.8), then passing to the limit as the original wing with its sharp edge is restored.

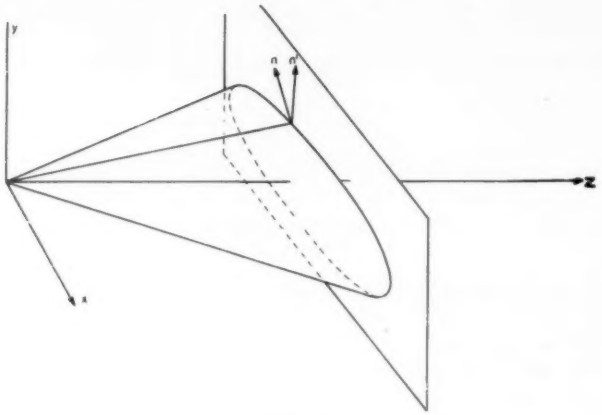

FIG. 13.

$$\frac{\partial \varphi}{\partial n'} = \frac{W_0}{2z}\frac{\partial (r^2)}{\partial n'} = \frac{W_0}{z}\left[x \cos (n'x) + y \sin (n'x)\right], \tag{5.8}$$

where (n, x) is the angle between the normal n' and the x-axis.

By superposition of conical flows and of solutions of other degrees of homogeneity, flows around a great variety of profiles may be obtained.

ON THE CHARACTERIZATION OF FIELDS OF DIABATIC FLOW*

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INTRODUCTION

In an earlier paper (Reference 1) the steady flow of a compressible fluid containing distributed heat sources was described in terms of the velocity vector **V** and alternately two other vectors **W** and **M**. For rotational flow the **W** language was found to lead to more simple differential equations than the other two languages. The physical content of the equations expressed in each of the three languages was, of course, the same. When each of the vector fields **V**, **W**, **M**, was in turn assumed to be irrotational, however, the corresponding fields of diabatic flow were radically different in their physical characteristics.

These results suggest several questions concerning the characterization of fields of steady diabatic flow:

(*i*) is **W** the most convenient vector to use in formulating the equations for rotational flow?

(*ii*) how do the restrictions upon the heat source function Q depend upon the character of the vector function **N** chosen to represent an irrotational field of flow?

(*iii*) can a relation be established between the character of a vector representation **N** of an irrotational flow field and the form of the partial differential equation for the potential function φ_N ?

We answer these questions by introducing into the flow equations a vector **N** proportional to **V**, the scalar proportionality factor (V/N) not at first being explicitly restricted. From the form alone of the derived equations in **N**, it is possible to answer the first question. The assumption of irrotationality of the **N** field, $\nabla \times \mathbf{N} = 0$, then leads to the required information about irrotational fields of flow.

Through proper choice of the **V**, **N** relation and subsequent inspection of the form of the **N** equations, a number of new types of irrotational diabatic flow can be discussed by giving their *character*, specified by the function $g(N) = V/(N^2RT)^{1/2}$ and its *form*, specified by an arbitrary function $F(\varphi_N)$ that enters the partial differential equation for the potential function φ_N . Four general types of flow are chosen for discussion to illustrate the wide variety of diabatic irrotational flows, as compared to adiabatic irrotational flows, that are possible even without explicit formulation of boundary conditions. We thereby emphasize the need for preliminary and simultaneous investigation of all steady diabatic flows at the formulational level before the attempt is made to construct special solutions.

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BASIC EQUATIONS

1. Derivation of N equations. We use essentially the same fundamental equations as in Reference 1, namely those governing the steady motion of an inviscid compressible fluid that contains heat sources.

$$\frac{1}{\rho}\nabla p + \mathbf{V}\cdot\nabla\mathbf{V} = 0, \tag{1.1}$$

$$\nabla\cdot\rho\mathbf{V} = 0, \tag{1.2}$$

$$p - R\rho T = 0, \tag{1.3}$$

$$c_p\mathbf{V}\cdot\nabla T_t = T\mathbf{V}\cdot\nabla S = Tc_p\mathbf{V}\cdot\nabla \log (Tp^{-(\gamma-1)/\gamma}) = Q. \tag{1.4}$$

The quantity Q is the heat added locally per unit mass of fluid and unit time. In Reference 1 these equations were studied in the **V**, **W**, and **M** languages, **W** and **M** being related to **V** by the equations

$$\mathbf{V} = V_t\mathbf{W} = a\mathbf{M} \tag{1.5}$$

in which $V_t = (2c_pT_t)^{1/2}$ is the limiting velocity corresponding to local stagnation temperature T_t and $a = (\gamma RT)^{1/2}$ will be called the local velocity of sound.* In the present paper we make the more general substitution

$$\mathbf{V} = (gRT)^{1/2}\mathbf{N} \tag{1.6}$$

which includes the **W** and **M** transformations when g is appropriately specialized. For the present g can be any function of the coordinates and other variables and will not be restricted until later. Both g and $N = |\mathbf{N}|$ are dimensionless.

Substitution of (1.6) in (1.1, 2, 4) and elimination of V and ρ gives

$$\nabla \log p + g\mathbf{N}\cdot\nabla\mathbf{N} + \frac{1}{2}g\mathbf{N}\mathbf{N}\cdot\nabla \log gT = 0, \tag{1.7}$$

$$\nabla\cdot\mathbf{N} + \mathbf{N}\cdot\nabla \log p + \frac{1}{2}\mathbf{N}\cdot\nabla \log (g/T) = 0, \tag{1.8}$$

$$-\frac{\gamma-1}{\gamma}\mathbf{N}\cdot\nabla \log p + \mathbf{N}\cdot\nabla \log T = Q/[c_pT(gRT)^{1/2}] = 2q_N\,. \tag{1.9}$$

In (1.9) we have defined a quantity q_N which represents in dimensionless form the local heating of the fluid. We adjoin the equation obtained from (1.7) by scalar multiplication of **N**

$$\mathbf{N}\cdot\nabla \log p + \frac{1}{2}N^2g\mathbf{N}\cdot\nabla \log T = -\frac{1}{2}N^2g\mathbf{N}\cdot\nabla \log g - g\mathbf{N}\cdot\nabla\frac{1}{2}N^2 \tag{1.10}$$

*In a gas undergoing dissociation or chemical reactions, the velocity of sound is a function of the local thermochemical properties of the gas.

and solve (1.9, 10) for $\mathbf{N}\cdot\nabla \log p$, $\mathbf{N}\cdot\nabla \log T$ in terms of $\mathbf{N}\cdot\nabla\,(N^2/2)$, $\mathbf{N}\cdot\nabla \log g$ and q_N :

$$\mathbf{N}\cdot\nabla \log p = -g\Big(\mathbf{N}\cdot\nabla \frac{1}{2}N^2 + \frac{1}{2}N^2\mathbf{N}\cdot\nabla \log g + N^2 q_N\Big)\Big/\Big(1 + \frac{\gamma-1}{2\gamma}N^2 g\Big), \quad (1.11)$$

$$\mathbf{N}\cdot\nabla \log T$$

$$= \Big[2q_N - \frac{\gamma-1}{\gamma}g\Big(\mathbf{N}\cdot\nabla \frac{1}{2}N^2 + \frac{1}{2}N^2\mathbf{N}\cdot\nabla \log g\Big)\Big]\Big/\Big(1 + \frac{\gamma-1}{2\gamma}N^2 g\Big). \quad (1.12)$$

Substitution in (1.7,8) gives the equations of motion and continuity in **N** language.

$$\nabla \log p = g\Big(1 + \frac{\gamma-1}{2\gamma}N^2 g\Big)^{-1}\Big(\frac{\gamma-1}{2\gamma}g\mathbf{N}\cdot\nabla \frac{1}{2}N^2 - \frac{1}{2}\mathbf{N}\cdot\nabla \log g - q_N\Big)\mathbf{N} \quad (1.13)$$

$$- g\nabla \frac{1}{2}N^2 + g\mathbf{N}\times(\nabla\times\mathbf{N}),$$

$$\nabla\cdot\mathbf{N} = \Big(1 + \frac{\gamma-1}{2\gamma}N^2 g\Big)^{-1}\Big[-\frac{1}{2}\Big(1 - \frac{N^2 g}{\gamma}\Big)\mathbf{N}\cdot\nabla \log g \quad (1.14)$$

$$+ \frac{\gamma+1}{2\gamma}g\mathbf{N}\cdot\nabla \frac{1}{2}N^2 + (1 + N^2 g)q_N\Big].$$

For completeness we list the expressions for $\mathbf{N}\cdot\nabla \log T_t$, $\mathbf{N}\cdot\nabla \log p_t$, quantities proportional to the rate of variation of stagnation temperature and of stagnation pressure in the direction of flow.

$$\Big(1 + \frac{\gamma-1}{2\gamma}N^2 g\Big)\mathbf{N}\cdot\nabla \log T_t = \frac{1}{c_p}\mathbf{N}\cdot\nabla S = 2q_N\,, \quad (1.15)$$

$$\mathbf{N}\cdot\nabla \log p_t = -N^2 g q_N\Big/\Big(1 + \frac{\gamma-1}{2\gamma}N^2 g\Big). \quad (1.16)$$

As we might expect, these variations are proportional to q_N and vanish for $q_N = 0$. Furthermore T_t increases and p_t decreases in the direction of flow for positive q_N . The decrease of p_t owing to heat addition is known as "momentum pressure drop" in one-dimensional theory (compare Eqs. (15) and (60) in Reference 2 and references cited there).

2. Significance of choice of the function g. We can now observe how the character of the function g affects the appearance of the **N** equation. The factor $[1 + (\gamma-1)N^2 g/2\gamma]$ appears in both Eqs. (1.13, 14) together with g, ∇g and q_N . If either g or q_N is an explicit function of T (cf. the special case of **V** language described in Sec. 3) then it may not be especially profitable to reduce the equations to the **N** form because T cannot be determined and eliminated from (1.13, 14) prior to complete solution of the set (1.1, 2, 3, 4) or the set (1.12, 13, 14, 15). If, on the other hand, g and q_N are functions of N, p and the coordinates x_i or even of the individual components N_i of **N** together

with p and the coordinates, one can regard (1.13, 14) as a set of four simultaneous non-linear partial differential equations in the four dependent variables $\mathbf{N}$, p.

We choose to study a less general function g, namely

$$g = g(N) \tag{2.1}$$

corresponding to the form for q_N

$$q_N = q_N(N_i , x_i). \tag{2.2}$$

When g and q_N are thus restricted, the $\mathbf{N}$ equation of motion (1.13) contains as dependent variables only $\mathbf{N}$ and p and the $\mathbf{N}$ equation of continuity (1.14) *only* $\mathbf{N}$. Equation (1.14) can then be rewritten in a more compact form which exhibits the "sources" proportional to $q\mathbf{N}$ in the $g\mathbf{N}/H_1$ field, letting $g' = dg/dN$,

$$\nabla \cdot H_1\mathbf{N} = (1 + N^2 g)\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1} H_1 q_N , \tag{2.3}$$

where

$$\log H_1(N) = -\frac{1}{2}\int \left[\frac{\gamma + 1}{\gamma} g - \left(1 - \frac{N^2 g}{\gamma}\right)\frac{g'}{Ng}\right]\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1} d\,\frac{1}{2} N^2. \tag{2.4}$$

Equation (2.1) includes both $\mathbf{W}$ and $\mathbf{M}$ languages for which g has the forms

$$g_W = \frac{2\gamma}{\gamma - 1}(1 - W^2)^{-1}, \qquad (N = W), \tag{2.5}$$

$$g_M = \gamma, \qquad (N = M). \tag{2.6}$$

Since $N^2 g = V^2/RT$ has the same value for all $g(N)$, the relations among V, N, W and M are

$$(V^2/RT) = N^2 g = \frac{2\gamma}{\gamma - 1} W^2(1 - W^2)^{-1} = \gamma M^2 \tag{2.7}$$

These equations are often used in the form

$$\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right) = (1 - W^2)^{-1} = \left(1 + \frac{\gamma - 1}{2} M^2\right). \tag{2.8}$$

It is only $\mathbf{W}$ language that a term in $\mathbf{N} \cdot \nabla N$ (or $\nabla \cdot \mathbf{N}$) does not appear in the equation of motion. The $\mathbf{W}$ language therefore yields more simple equations than any other $\mathbf{N}$ (or $\mathbf{V}$) language, and $\mathbf{W}$ is thus the most convenient vector to use in formulating the equations for rotational flow.

PROPERTIES OF IRROTATIONAL N FIELDS

3. Integrability of the equation of motion. We now suppose that the $\mathbf{N}$ field is continuous and irrotational and therefore admits a potential function φ_N

$$\mathbf{N} = \nabla \varphi_N . \tag{3.1}$$

The consequences of the assumption of irrotationality for an $\mathbf{N}$ field were first examined

in References 3a, b for the adiabatic **M** and **W** cases. In Reference 1 we saw that irrotational *diabatic* **M** and **W** flows were quite different from one another. We may therefore now expect a still wider variety of flows because of flexibility in choice of the function g.

With $\nabla \times \mathbf{N} = 0$, the equation of motion becomes

$$\nabla \log p + \eta \nabla \varphi_N + g \nabla \frac{1}{2} N^2 = 0 \tag{3.2}$$

in which

$$\begin{aligned} \eta &= \frac{1}{2} g \mathbf{N} \cdot \nabla \log (gT) \\ &= g\left(1 + \frac{\gamma - 1}{2\gamma} N^2 g\right)^{-1} \left[q_N - \frac{\gamma - 1}{2\gamma} g \mathbf{N} \cdot \nabla \frac{1}{2} N^2 + \frac{1}{2} \mathbf{N} \cdot \nabla \log g \right]. \end{aligned} \tag{3.3}$$

We distinguish two cases, $\eta = 0$ and $\eta \neq 0$. For the degenerate case $\eta = 0$ we suppose that g is not restricted to be a function of N alone. Then from (3.2, 3)

$$\nabla \log p + g \nabla \frac{1}{2} N^2 = 0, \tag{3.4}$$

$$\frac{\gamma - 1}{2\gamma} g \mathbf{N} \cdot \nabla \frac{1}{2} N^2 - \frac{1}{2} \mathbf{N} \cdot \nabla \log g = q_N . \tag{3.5}$$

Equation (3.4) implies that p and g are functions of N alone. Also Eq. (3.3) shows that (gT) and therefore, by (1.6), that (V/N) are constant on each streamline. This type of flow therefore corresponds to irrotational **V** flow (see Reference 1) that has been generalized to allow variation of (gT) between streamlines.

When $\eta \neq 0$, it is sufficient to consider only functions of N alone as was shown to be appropriate in Sec. 2. If the integrability condition with $g' = dg/dN$

$$\eta = \frac{1}{2} g \left[2 q_N + N \mathbf{N} \cdot \nabla N \left(\frac{g'}{Ng} - \frac{\gamma - 1}{\gamma} g \right) \right] \left(1 + \frac{\gamma - 1}{2\gamma} N^2 g \right)^{-1} = F(\varphi_N) \tag{3.6}$$

for the equation of motion (3.2) is satisfied then the latter integrates to

$$\log P_N + \int F(\varphi_N)\, d\varphi_N = \text{constant}, \tag{3.7}$$

where

$$P_N = p H_2(N) = p \exp \int g\, d\, \tfrac{1}{2} N^2. \tag{3.8}$$

(Integration of (3.3) also yields a relation among N, p and T along each streamline.) Equation (3.7) is the analogue of the barotropic condition $p = f(\rho)$ for irrotational **V** fields of flow. It states that P_N is constant on potential surfaces and expresses the variation of P_N along streamlines, i.e. $\partial \log P_N / \partial s = N F(\varphi_N)$. The function P_N becomes equal to the stagnation pressure $P_W = p_t$ in **W** language. In general the behavior of P_N is similar to p_t in that as N (or W) increases, p decreases for constant P_N or p_t. It is noted that in terms of H_2, the continuity equation (1.14) becomes (compare (2.3))

$$\nabla \cdot (g \mathbf{N} / H_2) = H_2^{-1} (1 + N^2 g) F(\varphi_N) \tag{3.9}$$

showing that "sources" in the $(g\mathbf{N}/H_2)$ field are proportional to $F(\varphi_N)$.

The integrability condition can be rewritten in two other useful forms by combining it with the continuity equation (1.14). Then

$$g^{-1}\left[\frac{\gamma+1}{\gamma}\,g-\left(1-\frac{N^2g}{\gamma}\right)\frac{g'}{Ng}\right]F(\varphi_N)$$
$$=\left(\frac{g'}{Ng}-\frac{\gamma-1}{\gamma}\,g\right)\nabla^2\varphi_N+2\left(g-\frac{g'}{Ng}\right)q_N \tag{3.10}$$

and

$$g^{-1}\left(1-\frac{N^2g}{\gamma}\right)\left(1+\frac{1}{2}\frac{Ng'}{g}\right)F(\varphi_N)$$
$$=-\frac{1}{2}N^3\left(\frac{g'}{Ng}-\frac{\gamma-1}{\gamma}\,g\right)\nabla\cdot\mathbf{s}+\left[1-N^2\left(g-\frac{g'}{Ng}\right)\right]q_N \tag{3.11}$$

in which $\mathbf{s}=\mathbf{N}/N$ is a unit vector in the direction of flow. According to Reference 3a,

$$\nabla\cdot\mathbf{s}=\lim_{\Delta A\to 0}\frac{1}{\Delta A}\frac{\partial\Delta A}{\partial s}$$

can be interpreted as the fractional rate of change of stream tube area ΔA with respect to arc length ds along the stream tube. Equations (3.6, **10, 11**) are alternative statements of the relation between the character of the function $g(N)$ and the mode of variation of $\nabla\cdot\mathbf{s}$ and q_N throughout the field of flow. Since the coefficients in these equations depend only upon N and $g(N)$, through special choice of the function $g(N)$ we will later be enabled to study flows characterized by special relations among the functions q_N, $\nabla\cdot\mathbf{s}$ and $F(\varphi_N)$.

4. The partial differential equation satisfied by the potential function φ_N. If $\mathbf{N}=\nabla\varphi_N$ is substituted directly into (1.14), the resulting equation contains q_N as well as derivatives of φ_N. Because in general the function q_N is not open to arbitrary specification in an irrotational $\mathbf{N}$ field, it is desirable to consider $F(\varphi_N)$ that appears in the integrability condition as the basic function describing the form of the flow. For a given $g(N)$ and $F(\varphi_N)$ the behavior of q_N follows from the integrability condition. It is therefore best to use Eq. (3.9) which does not contain q_N and which can be expanded to read

$$\nabla\cdot\mathbf{N}=g^{-1}(1+N^2g)F(\varphi_N)+\left(g-\frac{g'}{Ng}\right)\mathbf{N}\cdot\nabla\frac{1}{2}N^2. \tag{4.1}$$

Substitute

$$\nabla\cdot\mathbf{N}=\sum_i\frac{\partial^2\varphi_N}{\partial x_i^2},\qquad \mathbf{N}\cdot\nabla\frac{1}{2}N^2=\sum_i\left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\frac{\partial^2\varphi_N}{\partial x_i^2}+2\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j}.$$

The desired differential equation for the potential function is then

$$\sum_i\frac{\partial^2\varphi_N}{\partial x_i^2}\left[1-G\left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\right]-2G\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j}=\left(N^2+\frac{1}{g}\right)F(\varphi_N) \tag{4.2}$$

in which

$$G = g - \frac{g'}{Ng}. \tag{4.3}$$

The type of this partial differential equation is determined by the sign of the determinant $D = \det | \alpha_{ij} |$ where α_{ij} is the coefficient of $\partial^2\varphi_N/\partial x_i \partial x_j$ in the partial differential equation. Calculation of D gives

$$D = 1 - N^2G = 1 + N\frac{g'}{g} - N^2g. \tag{4.4}$$

The differential equation (4.2) is therefore of

$$\left.\begin{array}{l}\text{elliptic}\\ \text{parabolic}\\ \text{hyperbolic}\end{array}\right\} \quad \text{type for} \quad \begin{cases} G < N^{-2}, \\ G = N^{-2}, \\ G > N^{-2}. \end{cases} \tag{4.5}$$

Since G is a function of N alone, the equation can change type within the field of flow where N attains the values specified in (4.5). This behavior is similar to that of irrotational adiabatic flow in the **V** field which is of elliptic, parabolic or hyperbolic type for $M <, =, > 1$. There is however, a major difference. The character of the function $g(N)$ alone determines the values of N, if there be any, for which each of the expressions in (4.5) is satisfied. Appropriate choice of $g(N)$ can then lead to flows whose type is fixed (e.g. elliptic for all values of N) as well as to flows which change type for specified values of N. Because the character of g likewise determines the mutual behavior of $\nabla \cdot \mathbf{s}$, q_N and $F(\varphi_N)$ (cf. Eq. (3.11)), the required connection between character of g and behavior of q_N has thereby been demonstrated. Whenever, for example, $D = 0$, the relation between $\nabla \cdot \mathbf{s}$ and $F(\varphi_N)$ does not depend upon q_N (cf. Eq. 3.11). In a later section, we shall study some flows illustrating these remarks.

There may be a question at first sight as to whether $F(\varphi_N)$ and q_N can be found such that φ_N will satisfy both (3.10) and (4.2). We can analyze the situation as follows. Let $\mathbf{N} = [N_i]$, $i = 1, 2, 3$. For rotational flow there are five dependent variables N_i , p and q_N that satisfy the four equations (1.13, 14) giving an underdetermined system. The variation of one dependent variable, e.g. p, q_N (or one N_i) can then be specified arbitrarily. Now when $N_i = \partial\varphi_N/\partial x_i$ we have seven equations in the six variables N_i , p, q_N and φ_N and the system is overdetermined. The equation of motion (3.2) however reduces to the two equations (3.6, 7) which contain the new (arbitrary) function $F(\varphi_N)$. As this function is not determinable from the basic equation we regard its specification as part of the formulation of our problem and can consider then (3.7) to be an integral of the motion which eliminates p from further consideration. We now take (3.10) and (1.14 or 4.2) to form the basic system that must be solved. This system contains q_N , F_N , φ_N and its derivatives and is accordingly underdetermined. If we choose q_N to be the known function (1.14) with ($N = \nabla\varphi_N$) can presumably be solved for φ_N . There would then be, however, no reason to suppose that these functions φ_N and q_N would satisfy (3.10). We shall consequently regard (4.2) as the basic equation. Choice of the functions g and $F(\varphi_N)$ will specify the *character* and *form* of the flow respectively. Presumably specification of both form and character will work a corresponding restriction

upon the nature of the boundary conditions that can be imposed on the function φ_N. For example we have already seen that the type and change of type (with corresponding adjustment of the nature of the appropriate boundary conditions) of the partial differential equation (4.2) is entirely governed by the character of the flow.

The solution of a problem in diabatic irrotational flow can then be visualized as consisting of the following steps. With character and form of the flow specified together with matching boundary conditions, Eq. (4.2) can be solved for φ_N. Substitution of φ_N thus determined in the integrability condition (3.6 or 10) and the "Bernoulli" type equation (3.7) yields the functions q_N and p. The system consisting of Eqs. (1.13, 14) can then be regarded as solved. Determination of the temperature follows by integration of (1.12) if T is known at one point on each streamline. The velocity $\mathbf{V}$ and heat function Q can also be calculated subsequently from (1.6, 9).

The indirect nature of this solution process must be noted. Although one might normally expect Q to be known rather than $F(\varphi_N)$, it is the latter that, at present at least, must be chosen first and Q thereafter determined. It is this peculiarity of irrotational diabatic flow which leads us to investigate in general fashion irrotational flows whose character only is specified, in order that we may derive more explicit equations containing q_N.

THE RELATIONS AMONG CHARACTER, FORM, TYPE CHANGE AND HEAT SOURCES

5. Methods of investigation. We now propose to illustrate the remarks in the last section by examining a number of flows of given character. There are several paths we may follow in proceeding to this specialization. We could for example try to find all functions for which the flow remains of a given type. This procedure is simple only for cases for which D is constant throughout the flow. We might also return to Eq. (3.11), which connects the physically visualizable quantities $\nabla \cdot \mathbf{s}$ and q_N, and impose conditions on the coefficients appearing there in order to insure, for example, that $\nabla \cdot \mathbf{s}$ be proportional to $F(\varphi_N)$ throughout the field of flow. As a third possibility we could assume *a priori* convenient special forms for $g(N)$ (such as rational functions) and examine the corresponding characteristics of the flow. Any of these procedures will lead to conditions upon $g(N)$ and will supply partial answers to the questions implied by the other procedures.

For most of the special flows to be discussed now the first procedure is convenient. Let us suppose throughout the flow

$$D = D_0 \text{ constant.} \tag{5.1}$$

Accordingly as D_0 is $>$, $=$ or <0 we shall term the flow field elliptic, parabolic or hyperbolic. By (4.4) we are then led to the differential equation for $g(N)$

$$N \frac{g'}{g} - N^2 g + 1 - D_0 = 0, \tag{5.2}$$

whose solution for $D_0 \neq -1$ is

$$g = k^{-2}(1 + D_0)[(N/k)^{1-D_0} - (N/k)^2]^{-1} \tag{5.3}$$

and for $D_0 = -1$

$$g = N^{-2}(\log k - \log N)^{-1}. \tag{5.4}$$

In both cases, (and also for D not a constant)

$$G = \frac{1 - D_0}{N^2} \tag{5.5}$$

the partial differential equation for φ_N becomes

$$\sum_i \frac{\partial^2 \varphi_N}{\partial x_i^2}\left[N^2 - (1 - D_0)\left(\frac{\partial \varphi_N}{\partial x_i}\right)^2\right] - 2(1 - D_0)\sum_{i>j}\frac{\partial \varphi_N}{\partial x_i}\frac{\partial \varphi_N}{\partial x_j}\frac{\partial^2 \varphi_N}{\partial x_i\,\partial x_j} = N^2\left(N^2 + \frac{1}{g}\right)F(\varphi_N) \tag{5.6}$$

and the integrability condition in the form (3.11) simplifies to

$$g^{-1}\left(1 - \frac{N^2 g}{\gamma}\right)(1 + D_0 + N^2 g)F(\varphi_N) = 2D_0 q_N + N\left(1 - D_0 - \frac{N^2 g}{\gamma}\right)\nabla \cdot \mathbf{s}. \tag{5.7}$$

The flows characterized by (5.6) do not resemble "incompressible" adiabatic irrotational flows for small N unless $D_0 = 1$ (cf. Sec. 6) because all terms in the coefficient of $\partial^2 \varphi_N / \partial x_i \partial x_j$ are of order N^2. The Glauert-Prandtl type of approximation can be made, however, for suppose that throughout the flow

$$\frac{N_1}{N_0} \sim 1, \qquad \frac{N_2}{N_0} \sim \frac{N_3}{N_0} \sim 0,$$

where N is taken to differ but little from N_0 anywhere. Then (5.6) becomes

$$D_0 \frac{\partial^2 \varphi_N}{\partial x_1^2} + \frac{\partial^2 \varphi_N}{\partial x_2^2} + \frac{\partial^2 \varphi_N}{\partial x_3^2} = \left[N_0^2 + \frac{1}{g(N_0)}\right]F(\varphi_N). \tag{5.8}$$

That the value of D_0 determines the type of the differential equation is now quite clear. Equation (5.7) may be developed in similar approximate fashion once the value of D_0 is chosen, and the function H_2 (Eq. (3.8)) can be regarded as a function of N_0.

For some flows of interest (such as **W** and **M** flows) the type of the partial differential equation (4.2) may change in the flow field. A simple application of the second procedure is then appropriate. Let

$$g = g_0 + g_1 N + g_2 N^2 + \cdots. \tag{5.9}$$

Then for $N \ll 1$

$$G = g_0 + g_1 N - \frac{g_1 + 2g_2 N}{N(g_0 + g_1 N)} + \cdots. \tag{5.10}$$

Only for $g_0 \neq 0$ will Eq. (4.2) admit an "incompressible" approximation for $N \ll 1$,

$$\nabla^2 \varphi_N = g_0^{-1} F(\varphi_N). \tag{5.11}$$

The Glauert-Prandtl approximation can be written down for almost any choice of $g(N)$.

6. Elliptic flow with $D_0 = 1$. The general formula (5.3) for g corresponding to flow of constant type reduces for $D_0 = 1$ to $G = 0$ and

$$g = H_2 = 2/(k^2 - N^2). \tag{6.1}$$

The equation for φ_N (compare (2.3) and (3.10)) is

$$\nabla^2\varphi_N = \frac{1}{2}(k^2 + N^2)F(\varphi_N) \tag{6.2}$$

which in the Glauert-Prandtl approximation (cf. (5.8)) is

$$\nabla^2\varphi_N = \frac{1}{2}(k^2 + N_0^2)F(\varphi_N). \tag{6.3}$$

The heating parameter q_N is given by

$$q_N = \frac{1}{2}k^2\left(k^2 - \frac{\gamma + 2}{\gamma}N^2\right)(k^2 - N^2)^{-1}F(\varphi_N) + \frac{1}{\gamma}N^3(k^2 - N^2)^{-1}\nabla\cdot\mathbf{s}. \tag{6.4}$$

The "Bernoulli" equation becomes

$$\log\left(\frac{p}{k^2 - N^2}\right) + \int F(\varphi_N)\,d\varphi_N = \text{constant}. \tag{6.5}$$

Since the relation between W and N is

$$W^2 = (\gamma - 1)N^2/(\gamma k^2 - N^2); \tag{6.6}$$

values of $W^2 = 0$, $(\gamma - 1)/(\gamma + 1)$, 1 correspond to 0, $\gamma/(\gamma + 2)$, 1 for $(N/k)^2$.

It is noted that flow discontinuities such as stationary shocks and combustion fronts cannot occur within this or any other field wholly of elliptic type.

7. Parabolic flow: $D_0 = 0$. The only function g that gives wholly parabolic flow is

$$g = 1/N(k - N) \tag{7.1}$$

obtained from (5.3) where $D_0 = 0$. In this case $G = 1/N^2$, $H_2 = (k - N)^{-1} = Ng$. The partial differential equation for φ_N becomes

$$\sum_i\left[N^2 - \left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\right]\frac{\partial^2\varphi_N}{\partial x_i^2} - 2\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j} = kN^3F(\varphi_N) \tag{7.2}$$

of which the Glauert-Prandtl approximation is

$$\frac{\partial^2\varphi_N}{\partial x_2^2} + \frac{\partial^2\varphi_N}{\partial x_3^2} = kN_0F(\varphi_N). \tag{7.3}$$

When $D \equiv D_0 \equiv 0$ the coefficient of q_N in Eq. (3.11) vanishes and the equation expresses the proportionality of $\nabla\cdot\mathbf{s}$ and $F(\varphi_N)$ (cf. also 2.3)

$$\nabla\cdot\mathbf{s} = kF(\varphi_N). \tag{7.4}$$

The condition for vanishing of the coefficient of q_N is thus synonymous with the condition for wholly parabolic flow. Equation (7.4) expresses a simple geometrical property of parabolic flow, namely that the fractional rate of change of stream tube area in the direction of flow is a function of φ_N only. Accordingly, the fractional rate of change of

area of every stream tube intersecting a given potential surface is the same at that surface.

Because (3.11) and (7.4) do not contain q_N we use (3.10) to express its variation

$$q_N = \frac{1}{2} N(k - N)^{-1}\left(\frac{\gamma + 1}{\gamma} N^2 - 2kN + k^2\right)F(\varphi_N) + \frac{1}{2}(k - N)^{-1}\left(k - \frac{\gamma + 1}{\gamma} N\right)\nabla^2\varphi_N . \tag{7.5}$$

The "Bernoulli" equation becomes

$$\log \frac{p}{k - N} + \int F(\varphi_N)\, d\varphi_N = \text{constant}, \tag{7 6}$$

and the relation between W and N

$$W^2 = \frac{\gamma - 1}{2\gamma} N/\left(k - \frac{\gamma + 1}{2\gamma} N\right) \tag{7.7}$$

corresponding to $W^2 = 0, (\gamma - 1)/(\gamma + 1), 1$ for $(N/k) = 0, \gamma/(\gamma + 1), 1$.

8. Hyperbolic flow with $D_0 = -(\gamma + 1)$. The character of g which is sufficient if the coefficient of $F(\varphi_N)$ in (3.11) is to vanish is expressed as

$$g = \gamma/N^2. \tag{8.1}$$

This corresponds to a type of wholly hyperbolic flow specified by $D_0 = -(\gamma + 1)$, $k \to \infty$ and to $G = +(\gamma + 2)/N^2$, $H_2 = N^\gamma$. The equation for φ_N is

$$\sum_i \left[N^2 - (\gamma + 2)\left(\frac{\partial\varphi_N}{\partial x_i}\right)^2\right]\frac{\partial^2\varphi_N}{\partial x_i^2} - 2(\gamma + 2)\sum_{i>j}\frac{\partial\varphi_N}{\partial x_i}\frac{\partial\varphi_N}{\partial x_j}\frac{\partial^2\varphi_N}{\partial x_i\,\partial x_j} = \frac{\gamma + 1}{\gamma} N^4 F(\varphi_N) \tag{8.2}$$

or in the condensed form of (3.9)

$$\nabla\cdot(N^{-\gamma-2}\mathbf{N}) = \frac{\gamma + 1}{\gamma} N^{-\gamma} F(\varphi_N). \tag{8.3}$$

The Glauert-Prandtl approximation is

$$-(\gamma + 1)\frac{\partial^2\varphi_N}{\partial x_1^2} + \frac{\partial^2\varphi_N}{\partial x_2^2} + \frac{\partial^2\varphi_N}{\partial x_3^2} = \frac{\gamma + 1}{\gamma} N_0^2 F(\varphi_N). \tag{8.4}$$

Since the coefficient of $F(\varphi_N)$ in (3.11) is zero, q_N is given by the simple expression

$$q_N = \frac{1}{2} N\nabla\cdot\mathbf{s}. \tag{8.5}$$

In this case the area change of stream tubes is determined solely by q_N and N. The Bernoulli equation is

$$\log(pN^\gamma) + \int F(\varphi_N)\, d\varphi_N = \text{constant}, \tag{8.6}$$

and the relation between W and N

$$W^2 = \frac{\gamma - 1}{\gamma + 1}. \tag{8.7}$$

This particular wholly hyperbolic flow is thus characterized by a constant value of W (or $M = 1$) throughout the field of flow.

9. Mixed flow. It has been seen how proper choice of the character of the flow, as defined by the function $g(N)$, can establish flow whose type is constant throughout the field of flow. That the type may change within the flow field is illustrated by reconsideration of the **W** and **M** fields of irrotational diabatic flow (Reference 1). (The functions g_W and g_M are expressible in the form of (5.9) and accordingly admit the incompressible as well as the Glauert-Prandtl approximation; see Reference 1.) The partial differential equation for φ_M is of elliptic, parabolic or hyperbolic type accordingly as $M(=N) <, =,$ or $> (1/\gamma)^{1/2}$. The partial differential equation for φ_W changes type similarly at $W = N = [(\gamma - 1)/(\gamma + 1)]^{1/2}$, $(M = 1)$. It is the latter field which completes the set of three flows determined by the vanishing in turn of the coefficients of q_N, $F(\varphi_N)$ and $\nabla \cdot \mathbf{s}$ in equation (3.11), for an irrotational **W** field (Reference 1) possesses the property that

$$q_W = \frac{\gamma - 1}{2\gamma} F(\varphi_N) \tag{9.1}$$

10. Concluding remarks. If, in the definition $\mathbf{N} = \mathbf{V}[g(N)RT]^{-1/2}$, g is taken to be a function of N alone, any **N** language yields a simpler set of fundamental differential equations for rotational diabatic flow than the **V** language. In particular the **W** language (derived from $g = 2\gamma/(1 - N^2)(\gamma - 1)$) corresponds to the most simple expression of the equation of motion and hence is best suited to describe a general rotational diabatic flow.

With proper choice of the function $g(N)$, which is taken to determine the *character* of the flow, the partial differential equation for the potential function φ_N of irrotational **N** flow can be made wholly of one type (e.g. elliptic) or of mixed type throughout the field of flow. The differential equation for φ_N always contains an arbitrary function $F(\varphi_N)$ which fixes the *form* of the flow and also occurs in the analogue of the Bernoulli equation. Choice of the flow *character* alone leads to the form of the relation among $F(\varphi_N)$, $\nabla \cdot \mathbf{s}$ (the fractional rate of change of stream tube area in the direction of flow) and q_N (proportional to the local rate of heat addition per unit mass of fluid). Because of the universal occurrence of $F(\varphi_N)$ in the differential equations for φ_N there is need for further study of such quasi-linear partial differential equations containing arbitrary functions of the dependent variable if irrotational diabatic flows are to be characterized more fully with any degree of generality.

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ON THE PLASTIC BENDING OF CIRCULAR PLATES UNDER UNIFORM TRANSVERSE LOADS*

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1. Introduction. The primary purpose of this paper is to investigate the mechanical behaviour of thin plates subjected to distributed transverse loads which produce plastic deformations. Throughout this discussion the plates will be considered as being homogeneous and composed of isotropic materials exhibiting a gradual transition from the elastic to the plastic state. Two alternative theories of plasticity, namely a theory of plastic flow and a theory of plastic deformation,[1] will be used, and their predictions will be compared with each other. In particular, the case of a circular plate subjected to a uniformly distributed transverse load will be considered in detail.

As applied to strain-hardening materials, the theory of plastic flow is based on the assumption that the rate of change of strain is uniquely determined by the existing stress and the rate of change of stress. Stress-strain relations of this type have been considered by Prager [2, 3] and, more recently, by Handelman, Lin, and Prager [4]; they may be considered as a natural generalization of the stress-strain relations proposed for perfectly plastic materials by Prandtl [5] and Reuss [6]. On the other hand the theory of plastic deformation developed by Hencky [7] for perfectly plastic materials and applied by Nadai [8, p. 75] to materials with continuous transition from the elastic to the plastic state is based on the assumption that in the plastic state as in the elastic state, the state of strain is uniquely determined by the state of stress. These stress-strain relations [9] are simpler than those of the flow theory, and thus have received more consideration in mathematical studies concerning permanent deformations in structural elements.

Until recently, comparatively little attention had been given to the plastic bending of plates; however, some notable contributions in this field have recently been made by Russian investigators, in particular, Ilyushin [10] and Sokolovsky [11]. Most of their results, however, do not apply to materials which exhibit gradual transition from the elastic to the plastic state.

2. Stress-strain relations. Investigation of the bending of plates made of a strain-hardening material by means of the theory of plastic flow is extremely difficult if the customary form of the stress-strain relations for such materials is adopted [4, Eq. 27]. Indeed, from this form of the stress-strain relations, the rate of change of the stress tensor can not be obtained explicitly in terms of the rate of change of the strain tensor, the strain tensor itself, and the strain invariants, as is necessary in the study of the bending of plates. This difficulty is avoided if slightly modified stress-strain relations for loading are adopted. Let the tensors of stress and strain be denoted by p_{ij} and ϵ_{ij}, their deviations by

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[1]This terminology is due to A. A. Ilyushin [1] (numbers in square brackets refer to the bibliography at the end of the paper).

$$r_{ij} = \rho_{ij} - \frac{1}{3}\rho_{kk}\delta_{ij}\,, \qquad e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij}$$

(δ_{ij} = Kronecker delta), and the elastic shear modulus and the *tangent shear modulus* by G_0 and G, respectively, the latter being assumed as a function of the *strain intensity*

$$E = \frac{2}{3}e_{ij}e_{ij}\,.$$

The stress-strain relation of the flow theory used in this paper then has the form

$$r^*_{ij} = 2G_0 e^*_{ij} - (G_0 - G)\frac{E^*}{E}e_{ij}\,. \tag{2.1}$$

Throughout the following discussion, the Latin subscripts run from 1 through 3, and rates of change of variables with respect to time will be indicated by asterisks. Stress-strain relations of the type (2.1) may be termed *stress theories* of plastic flow, while relations such as are treated in [4] may be called *strain theories* of plastic flow.

The expression (2.1) constitutes five linearly independent stress-strain relations which take into account only the change of shape of a deformed body. To these, a sixth expression dealing with the change in volume must be added. For the purpose of simplification, we shall assume the material to be incompressible:

$$\epsilon_{kk} = 0. \tag{2.2}$$

(Here, repeated indices indicate summation in accordance with the customary summation convention of tensor calculus.) Accordingly, Poisson's ratio $\nu = 1/2$, $\epsilon_{ij} \equiv e_{ij}$, and Young's modulus equals three times the shear modulus G_0.

In order to determine the quantity G which is a function of the strain intensity E only and thus independent of the specific type of loading, it is sufficient to consider a simple state of stress, e.g., simple tension. Indicating the tensile stress by ρ and the corresponding unit extension by ϵ, Eq. (2.1) reduces to the single equation

$$\frac{d\rho}{d\epsilon} = 3G, \tag{2.3}$$

and the strain intensity becomes $E = \epsilon^2$. Hence the *tangent shear modulus* G for any material obeying (2.1) can be determined if the stress-strain diagram for simple tension is known. For the material being considered here, the function $\rho = \rho(\epsilon)$ can be represented with sufficient accuracy by the power series,

$$\rho = a_1\epsilon + a_3\epsilon^3 + a_5\epsilon^5 + \cdots\,, \tag{2.4}$$

where a_1, a_3, a_5, etc. are constants. Thus the tangent shear modulus is given by

$$3G = a_1 + 3a_3E + 5a_5E^2 + \cdots\,, \tag{2.5}$$

where a_1 equals Young's modulus, i.e., $a_1 = 3G_0$.

It is more convenient from this point on to work with non-dimensional quantities. Accordingly, the following non-dimensional components of stress and stress deviation will be introduced: $\sigma_{ij} = \rho_{ij}/G_0$, $s_{ij} = r_{ij}/G_0$. By substituting these quantities and the series expansion (2.5) into Eq. (2.1), we obtain a stress-strain relation for the stress theory of plastic flow for a material with a stress-strain curve in simple tension which is represented by the series (2.4), or by the equivalent general stress-strain relation

$$s_{ij}^* = 2e_{ij}^* - [b_1 + b_2E + b_3E^2 + \cdots]E^*e_{ij}\,, \tag{2.6}$$

where

$$b_n = -(2n+1)a_{2n+1}/3G_0\,, \qquad n = 1, 2, 3, \cdots$$

The stress-strain relation of the theory of plastic deformation is of the form

$$r_{ij} = 2G^{\wedge}e_{ij}\,, \tag{2.7}$$

where $G^{\wedge} = G^{\wedge}(E)$ is the *secant shear modulus*. For simple tension Eq. (2.7) reduces to the single expression

$$\rho/\epsilon = 3G^{\wedge}.$$

Using the same series expansion (2.4) as in the plastic flow theory, we obtain

$$3G^{\wedge} = a_1 + a_3E + a_5E^2 + \cdots \tag{2.8}$$

Substituting this into Eq. (2.7) and using the non-dimensional form, we find

$$s_{ij} = 2e_{ij} - [c_1 + c_2E + c_3E^2 + \cdots]Ee_{ij}\,, \tag{2.9}$$

where

$$c_n = -2a_{2n+1}/3G_0\,.$$

3. Basic differential equations. We consider a thin plate subjected to distributed transverse loads, and assume that the resulting deflections are small in comparison to the thickness of the plate. We introduce the rectangular coordinates X_i, the X_1, X_2 plane coinciding with the plane horizontal middle surface of the undeformed plate, and the axis of X_3 being directed vertically downward. As is usual in the theory of thin plates, the stress components σ_{13}, σ_{23}, σ_{33} will be treated as small in comparison with the stresses parallel to the middle surface of the plate (σ_{11}, σ_{22} and σ_{12}). Restricting Greek subscripts to values 1 and 2, we may, therefore, write Eq. (2.6) in the form

$$\sigma_{\alpha\beta}^* = 2\{e_{\alpha\beta}^* + \delta_{\alpha\beta}e_{\gamma\gamma}^*\} - \{e_{\alpha\beta} + \delta_{\alpha\beta}e_{\gamma\gamma}\}[b_1 + b_2E + \cdots]E^*, \tag{3.1}$$

where $E = 2/3(e_{\alpha\beta}e_{\alpha\beta} + e_{\alpha\alpha}e_{\beta\beta})$.

We now adopt the following notations:

$$x_\alpha = X_\alpha/R_0\,, \qquad x_3 = 2X_3/h,$$

$$w = Wh/2R_0^2\,, \qquad M_{\alpha\beta} = 4\mathfrak{M}_{\alpha\beta}/h^2G_0\,, \tag{3.2}$$

$$l = 4qR_0^2/h^2G_0\,,$$

where R_0 is a fixed suitably chosen length, h the constant thickness of the plate, $W = W(X_\alpha)$ the deflection, q the intensity of the load, and $\mathfrak{M}_{\alpha\beta}$ are the flexural and twisting moments ($\mathfrak{M}_{\alpha\beta} = \int_{-h/2}^{h/2} X_3\rho_{\alpha\beta}\,dX_3$).

It is a fundamental hypothesis in the theory of elastic plates that points on a normal to the middle surface of the plate before deformation remain on a normal to the deformed middle surface. It is assumed that this hypothesis remains valid for the plastic materials considered here. Accordingly, the relation between the strains $\epsilon_{\alpha\beta}$ and the deflection W of the middle surface is [12, p. 323]

$$\epsilon_{\alpha\beta} = -X_3W_{,\alpha\beta}\,,$$

where $W_{,\alpha\beta} = \partial^2W/\partial X_\alpha\partial X_\beta$.

In non-dimensional form this expression becomes

$$\epsilon_{\alpha\beta} = -x_3 w_{,\alpha\beta} . \tag{3.3}$$

In terms of the deflection w, Eq. (3.1) then becomes

$$\sigma^*_{\alpha\beta} = -2x_3\Omega^*_{\alpha\beta} + \frac{2}{3} b_1 x_3^3 E_0^* \Omega_{\alpha\beta} + \frac{4}{9} b_2 x_3^5 E_0 E_0^* \Omega_{\alpha\beta} + \cdots , \tag{3.4}$$

where the quantity E_0 , related to E by means of $E = 2/3\ x_3^2 E_0$, is written as

$$\begin{aligned} E_0 &= \Omega_{\alpha\beta} w_{,\alpha\beta} , \\ \Omega_{\alpha\beta} &= w_{,\alpha\beta} + \delta_{\alpha\beta} w_{,\gamma\gamma} . \end{aligned} \tag{3.5}$$

The time rate of change of the non-dimensional bending moments

$$M^*_{\alpha\beta} = \int_{-1}^{1} x_3 \sigma^*_{\alpha\beta}\, dx_3$$

can now be easily calculated. We obtain

$$M^*_{\alpha\beta} = -\frac{4}{3}\left[\Omega^*_{\alpha\beta} - \frac{1}{5} b_1 E_0^* \Omega_{\alpha\beta} - \frac{2}{21} b_2 E_0^* E_0 \Omega_{\alpha\beta} - \cdots\right]. \tag{3.6}$$

For transversely loaded plates with small deflections, the equation of equilibrium[2] [13, p. 87] becomes

$$M^*_{\alpha\beta,\alpha\beta} + l^* = 0 \tag{3.7}$$

which, together with Eq. (3.6), furnishes

$$\Omega^*_{\alpha\beta,\alpha\beta} - \frac{1}{5} b_1 \{E_0^* \Omega_{\alpha\beta}\}_{,\alpha\beta} - \frac{2}{21} b_2 \{E_0^* E_0 \Omega_{\alpha\beta}\}_{,\alpha\beta} - \cdots = \frac{3}{4} l^* . \tag{3.8}$$

Finally, referring to definitions (3.5) and carrying out the necessary steps, we arrive at the basic non-linear partial differential equation in w for the flow theory, namely

$$\begin{aligned} & w^*_{,\alpha\alpha\beta\beta} - \frac{b_1}{5}\{w_{,\alpha\beta}\, w_{,\gamma\delta}\, w^*_{,\alpha\beta\gamma\delta} + 2w_{,\alpha\alpha}\, w_{,\gamma\delta}\, w^*_{,\beta\beta\gamma\delta} + w_{,\alpha\alpha}\, w_{,\beta\beta}\, w^*_{,\gamma\gamma\delta\delta} \\ & + 2w_{,\alpha\beta}\, w_{,\alpha\gamma\delta}\, w^*_{,\beta\gamma\delta} + 2w_{,\alpha\alpha}\, w_{,\beta\gamma\delta}\, w^*_{,\beta\gamma\delta} + 2w_{,\alpha\beta}\, w_{,\alpha\gamma\gamma}\, w^*_{,\beta\delta\delta} \\ & + 6w_{,\alpha\alpha}\, w_{,\beta\gamma\gamma}\, w^*_{,\beta\delta\delta} + 4w_{,\alpha\beta}\, w_{,\gamma\gamma\delta}\, w^*_{,\alpha\beta\delta} + w_{,\alpha\beta}\, w_{,\alpha\beta\gamma\delta}\, w^*_{,\gamma\delta} \\ & + w_{,\alpha\alpha}\, w_{,\beta\beta\gamma\delta}\, w^*_{,\gamma\delta} + w_{,\alpha\beta}\, w_{,\alpha\beta\gamma\gamma}\, w^*_{,\delta\delta} + 3w_{,\alpha\alpha}\, w_{,\beta\beta\gamma\gamma}\, w^*_{,\delta\delta} \\ & + 4w_{,\alpha\beta\beta}\, w_{,\alpha\gamma\delta}\, w^*_{,\gamma\delta} + 4w_{\alpha\beta\beta,}\, w_{,\alpha\gamma\gamma}\, w^*_{,\delta\delta} + 2w_{,\alpha\beta}\, w_{,\gamma\gamma\delta\delta}\, w^*_{,\alpha\beta}\} \\ & - \frac{b_2}{21}\{E_0^* E_0 \Omega_{\alpha\beta}\}_{,\alpha\beta} - \cdots = \frac{3}{8} l^* . \end{aligned} \tag{3.9}$$

The corresponding basic partial differential equation for the plastic deformation theory can be obtained in a similar manner. We note merely the final equation,

[2] See Eq. (g): $\mathfrak{M}_{11} = M_x$, $\mathfrak{M}_{22} = M_y$, $\mathfrak{M}_{12} = -M_{xy}$, $\mathfrak{M}_{21} = M_{yx}$.

$$w_{,\alpha\alpha\beta\beta} - \frac{c_1}{5}\{w_{,\alpha\beta}\, w_{,\gamma\delta}\, w_{,\alpha\beta\gamma\delta} + 2w_{,\alpha\alpha}\, w_{,\beta\gamma}\, w_{,\beta\gamma\delta\delta} + 2w_{,\alpha\alpha}\, w_{,\beta\beta}\, w_{,\gamma\gamma\delta\delta}$$

$$+ w_{,\alpha\beta}\, w_{,\alpha\beta}\, w_{,\gamma\gamma\delta\delta} + w_{,\alpha\beta}\, w_{,\alpha\gamma\delta}\, w_{,\beta\gamma\delta} + w_{,\alpha\beta}\, w_{,\alpha\gamma\gamma}\, w_{,\beta\delta\delta} \tag{3.10}$$

$$+ 4w_{,\alpha\beta}\, w_{,\alpha\beta\gamma}\, w_{,\gamma\delta\delta} + w_{,\alpha\alpha}\, w_{,\beta\gamma\delta}\, w_{,\beta\gamma\delta} + 5w_{,\alpha\alpha}\, w_{,\beta\gamma\gamma}\, w_{,\beta\delta\delta}\}$$

$$- \frac{c_2}{21}\{E_0^* E_0 \Omega_{\alpha\beta}\}_{,\alpha\beta} - \cdots = \frac{3}{8} l.$$

4. Example. We are now in a position to consider a specific problem in detail, namely the case where

a) the thin plate is made of 24 S-T aluminum alloy,

b) the contour of the plate is circular,

c) the plate is built-in along its entire boundary, and

d) the lateral load is uniformly distributed.

In Fig. 1 the graph (full curve) of the stress ρ and the corresponding unit extension ϵ is given for a tension test with a specimen of 24 S-T aluminum. The analytic approximation for this curve most suitable for our purposes is

$$\rho = 3G_0\left(\epsilon - \frac{10^6}{128}\epsilon^3\right), \tag{4.1}$$

where

$$3G_0 = 10.667 \times 10^6 \text{ lbs./sq. in.}$$

This relation is represented by the dashed curve in Fig. 1. Referring to the expansion (2.4), we find that the values of the coefficients for this material become

$$a_1 = 3G_0\,, \qquad a_3 = -\frac{3\cdot 10^6}{128} G_0\,, \qquad a_5 = a_7 = \cdots = 0.$$

From this it follows (see Eqs. (2.6) and (2.9)) that

$$b_1 = \frac{3\cdot 10^6}{128}, \qquad b_2 = b_3 = \cdots = 0 \tag{4.2}$$

and

$$c_1 = \frac{2\cdot 10^6}{128}, \qquad c_2 = c_3 = \cdots = 0.$$

With these values, the basic equations (3.9) and (3.10) are greatly simplified.

Since the load intensity is assumed to be constant, i.e., $l \equiv l_0$, the deflection w is independent of the polar angle θ. The transformation of Eq. (3.9) with the values (4.2) into the more suitable polar coordinate system gives a differential equation in w, now a function of the single non-dimensional radial coordinate $r = R/R_0$, where R is the dimensional radial coordinate and R_0 the outer radius. The actual transformation is quite laborious but straightforward, and for that reason it will be omitted here. Expressing the differentiation with respect to r by primes, we find

$$w^{*\text{iv}} + \frac{2}{r} w^{*\prime\prime\prime} - \frac{1}{r^2} w^{*\prime\prime} + \frac{1}{r^3} w^{*\prime} - \frac{b_1}{5} F = \frac{3}{8} l_0^*\,, \tag{4.3}$$

where $F = F(r, t, w', w'', w''', w^{iv}, w^{*'}, w^{*''}, w^{*'''}, w^{*iv})$ represents 24 terms and $l_0 = l_0(t)$.

The complete representation of Eq. (4.3) shows that this expression is homogeneous in the time rates of change, and therefore the time variable t can be changed to any

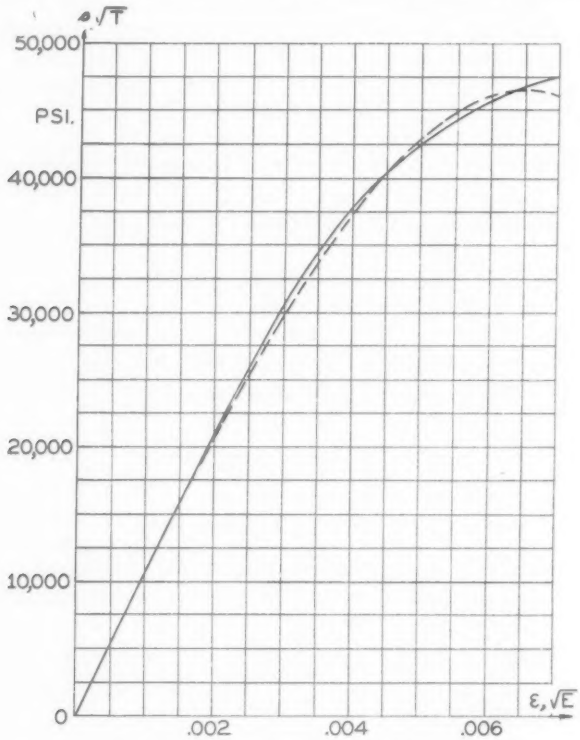

FIG. 1. Stress-strain diagrams: (1) 24S-T Aluminum Alloy (full curve); (2) Approximation as given by Eq. (4.1) (dashed curve).

other monotonically increasing variable. Since we assume the load to increase with time, t can be replaced by l_0, thus, $w^* = l_0^*(dw/dl_0)$.

It is found that if Eq. (4.3) is multiplied by the factor r then the differential equation can be integrated to a lower order expression, namely

$$rw^{*'''} + w^{*''} - \frac{1}{r}w^{*'} - \frac{b_1}{5}\left\{\left(4rw''w'' + 4w'w'' + \frac{1}{r}w'w'\right)w^{*'''}\right.$$

$$+ \left(8rw''w''' + 8w''w'' + 4w'w''' + \frac{2}{r}w'w'' - \frac{1}{r^2}w'w'\right)w^{*''} \qquad (4.4)$$

$$\left. + \left(4w''w''' + \frac{4}{r}w''w'' + \frac{5}{r}w'w''' - \frac{5}{r^2}w'w'' - \frac{8}{r^3}w'w'\right)w^{*'}\right\} = \frac{3}{16}r^2$$

which serves as the basic differential equation of a circular plate under uniform load for the plastic flow theory.

The corresponding basic differential equation for the plastic deformation theory of a circular plate under uniform load can be derived in a similar manner; one obtains

$$rw''' + w'' - \frac{1}{r}w' - \frac{c_1}{5}\left\{6rw''w''w''' + 6w'w''w''' + \frac{3}{r}w'w'w''' + 4w''w''w'' + \frac{3}{r}w'w''w'' - \frac{3}{r^2}w'w'w'' - \frac{4}{r^3}w'w'w'\right\} = \frac{3}{16}l_0r^2. \tag{4.5}$$

The problem of solving the non-linear differential equations (4.4) and (4.5) for $w(r, l_0)$ presents great mathematical difficulties, and there is little hope for obtaining solutions in closed form. We therefore will attempt to approximate the solutions by means of series of the following type

$$w(r, l_0) = w_1(r)l_0 + w_3(r)l_0^3 + w_5(r)l_0^5 + \cdots, \tag{4.6}$$

where the coefficients w_1, w_3, w_5, etc. of the odd powers of the parameter l_0 depend on r alone. Substituting this expansion into Eq. (4.4), and collecting terms involving like powers in l_0, we obtain a sequence of differential equations in the unknown functions w_1, w_3, w_5, $\cdots$. The first of these is

$$rw_1''' + w_1'' - \frac{1}{r}w_1' = \frac{3}{16}r^2 \tag{4.7}$$

or

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_1}{dr}\right)\right] = \frac{3}{16}r^2;$$

the next contains w_1 and w_3; only w_3 is unknown, however, since w_1 can be determined from Eq. (4.7). This procedure can be continued until the desired number of coefficients in (4.6) are found.

In view of the boundary conditions for a built-in circular plate,

$$w(1, l_0) = 0, \qquad w'(1, l_0) = 0, \tag{4.8}$$

and the requirement that $w(0, l_0)$ be finite, the solution of Eq. (4.7) is found to be

$$w_1 = \frac{3}{512}(r^2 - 1)^2; \tag{4.9}$$

this is nothing but the elastic solution. The differential equation for $w_3(r)$ then becomes

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_3}{dr}\right)\right] = \frac{72b_1}{5(128)^3}\{143r^6 - 129r^4 + 30r^2\}, \tag{4.10}$$

and that for w_5

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_5}{dr}\right)\right] = \frac{8\cdot 3^5\cdot 10^{11}}{25(128)^7}\{350{,}493r^{10} - 622{,}668r^8 + 419{,}112r^6 - 128{,}067r^4 + 15{,}270r^2\}. \tag{4.11}$$

Integrating these equations and substituting the results into the series (4.6), we obtain

$$w(r, l_0) = \frac{3}{512}(r^2 - 1)^2 l_0 + \frac{45}{4(12.8)^4}\{143r^8 - 344r^6 + 360r^4$$

$$- 260r^2 + 101\}l_0^3 + \frac{324}{(12.8)^7}\{584{,}155r^{12} - 1{,}868{,}004r^{10} \qquad (4.12)$$

$$+ 2{,}619{,}450r^8 - 2{,}134{,}450r^6 + 1{,}145{,}250r^4$$

$$- 529{,}860r^2 + 183{,}459\}l_0^5 + \cdots .$$

This equation is the series development up to terms of the 5th order in l_0 of the solution $w(r, l_0)$ to the differential equation (4.4) of the plastic flow theory.

Beginning with the differential Eq. (4.5), a similar expansion can be obtained for the deformation theory. The functions $w_1(r)$ and $w_3(r)$ in the series solution as given by the deformation theory are found to be identical with the corresponding functions obtained for the flow theory. This property can be shown to exist even under more general conditions; however, this fact will not be proved here. The first difference between the two theories occurs for the function w_5 . Up to terms of this order, the series representation of the deflection furnished by the deformation theory is found to be:

$$w(r, l_0) = \frac{3}{512}(r^2 - 1)^2 l_0 + \frac{45}{4(12.8)^4}\{143r^8 - 344r^6 + 360r^4$$

$$- 260r^2 + 101\}l_0^3 + \frac{540}{(12.8)^7}\{352{,}495r^{12} - 1{,}130{,}796r^{10} \qquad (4.13)$$

$$+ 1{,}591{,}650r^8 - 1{,}300{,}750r^6 + 697{,}500r^4$$

$$- 320{,}340r^2 + 110{,}241\}l_0^5 + \cdots .$$

The stress distribution throughout the circular plate can be calculated in the case of the plastic flow theory by transforming expression (3.4) into polar coordinates, substituting Eq. (4.12), and then integrating. The resulting non-dimensional expressions for the radial stress component P and the circumferential stress component Q together with those similarly obtained for the plastic deformation theory are valid for all values of $l_0 \leq l_0'$, where l_0' represents the maximum load which our assumed stress-strain law entitles us to consider. In order to calculate this load we return to Eq. (4.1). For uniaxial stress l_0' would correspond to the maximum height of the dashed curve in Fig. 1; for the combined state of stress we are considering, the stress-strain curve expresses the relation between the intensities of stress and strain, where the stress intensity

$$T = 3/2\, r_{ij}r_{ij} ,$$

reduces for simple tension to $T = \rho^2$. We have already seen that $E = 2/3\, e_{ij}e_{ij}$ reduces to ϵ^2 for simple tension. Thus, for combined states of stress, the coordinate axes of the dashed curve of Fig. 1 are labeled $T^{1/2}$ and $E^{1/2}$.

The non-dimensional stress intensity $S = T/G_0^2 = 3/2\, s_{ij}s_{ij}$ for the example being considered here is

$$S = P^2 + Q^2 - PQ.$$

Since the maximum value of $S^{1/2}$ occurs at the boundary, i.e., $x_3 = \pm 1$, $r = 1$, we find that for $l_0 = l_0' = .09$ a value $S^{1/2} = .01312$ is reached which takes us to the top of the stress-strain intensity curve in Fig. 1.

A review of the stress distribution on the upper surface of the plate shows that for $l_0 = .02$, see Fig. 2, there exists no appreciable difference between the data as given

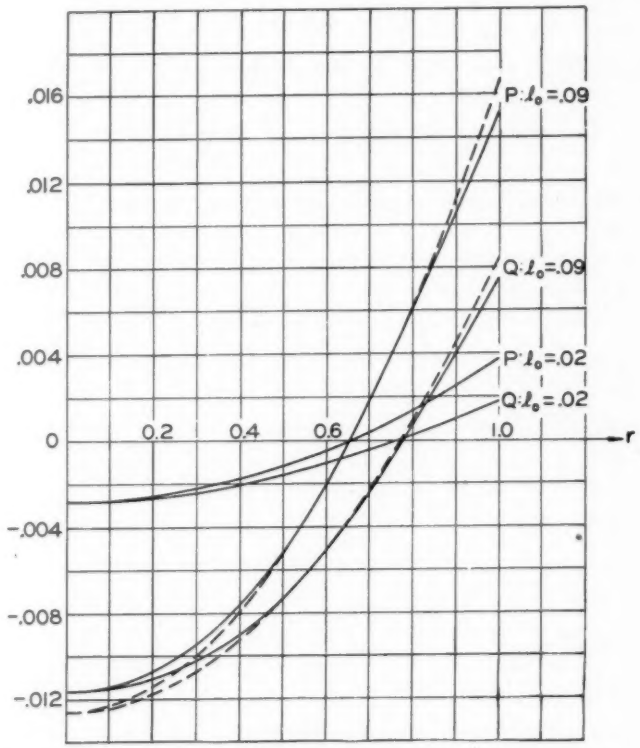

FIG. 2. Curves for the non-dimensional radial and circumferential stresses in the upper surface of the circular plate for $l_0 = .02$ and $l_0 = .09$ as given by the plastic theories (full curves) and the elastic theory (dashed curves).

by the elastic theory and the two plastic theories; to within five significant decimal places, the third order terms in l_0 have a negligible influence, and those of the fifth order have no influence at all. As the load increases to $l_0 = .04$, there is a slight deviation which becomes more and more pronounced as l_0 increases; however, to within five decimal places there is at no time a difference between the values given by the theories of plastic flow and of plastic deformation. For $l_0 = .06$ the third order terms have an influence ranging to 3.46% of the first order terms, while the fifth order terms reach only .53% of the first order terms. Third order terms range to an influence of 7.8%, and fifth order to 2.5% for the ultimate load $l_0 = l_0' = .09$. From this it is estimated that the solutions given by (4.12) and (4.13) approximate the real solutions with sufficient accuracy.

In order to obtain the work D done on the plate we calculate the following integral:

$$D = 2\pi \int_0^{R_0} \int_0^t q_0 \frac{\partial W}{\partial t}\, dt\, R\, dR$$

$$= 2\pi \int_0^{R_0} \left[q_0 W - \int_0^t W \frac{\partial q_0}{\partial t}\, dt \right] R\, dR.$$

In non-dimensional form—see (3.2)—the above reduces to

$$D = \pi G_0 h R_0^2 \int_0^1 \left[l_0 w - \int_0^t w \frac{\partial l_0}{\partial t}\, dt \right] r\, dr$$

or by changing the time parameter again to l_0 we obtain

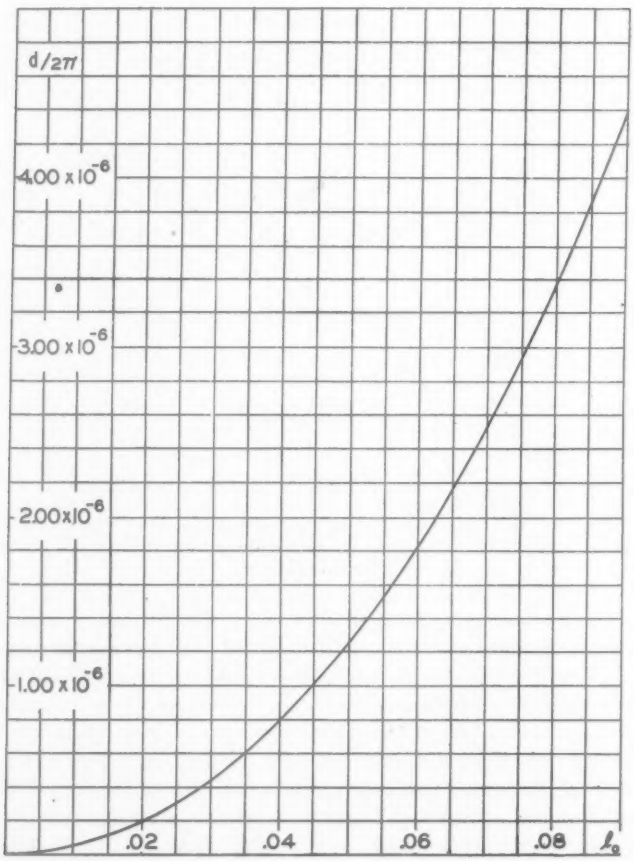

FIG. 3. Relation between non-dimensional loads l_0 versus non-dimensional work d as given by Eq. (4.14).

$$D = \frac{1}{2} G_0 h R_0^2 \, d,$$

where the non-dimensional work d is given by

$$d = 2\pi \int_0^1 \left[l_0 w - \int_0^{l_0} w \, dl_0 \right] r \, dr. \tag{4.14}$$

A graph of $d/2\pi$ versus l_0 is given in Fig. 3. We note that in the purely elastic case, where w reduces to simply the first term of (4.12) or (4.13), the above formula (4.14) reduces to the familiar form

$$d = \pi \int_0^1 l_0 \, wr \, dr.$$

In conclusion we can state that though the theories of plastic flow and plastic deformation are founded on different hypotheses their predictions for circular plates under uniform loads and for the specific material considered here are identical for all practical purposes. Both plastic theories however, give a deviation from the elastic solution ranging to better than 10% depending on the load and the material point under consideration.

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A MATHEMATICAL FORMULATION OF THE GENERAL CONTINUOUS DEFORMATION PROBLEM*

BY

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1. Introduction. The purpose of this paper is to formulate precisely and simply the general stress-strain problem for continuous flow or continuous deformation of continuous media such as viscous liquids or elastic or plastic solids. We shall use an *embedded coordinate system* which is considered to be embedded in the material so that each particle of the body carries with itself coordinates $x^i = (x^1, x^2, x^3)$ which[1] vary continuously from particle to particle but which do not vary with time for a given particle as the body deforms.

The square of the arc length for an embedded element $dx^i = (dx^1, dx^2, dx^3)$ is given by a quadratic form $ds^2 = g_{ij}\, dx^i\, dx^j$, summed[2] on $i = 1, 2, 3$ and $j = 1, 2, 3$, where the fundamental tensor[3] g_{ij} is a continuous function not only of x^i but also of the time t. This changing fundamental tensor is of basic importance in the theory and will be called the *deformation tensor*.

In terms of the deformation tensor the following two principles may be stated:

1. *Every continuous flow or continuous deformation may be described by the deformation tensor $g_{ij}(x^i, t)$ where g_{ij} is the fundamental tensor of space referred to embedded coordinates in the body.*

2. *All equations relating stress and finite strain must be expressible as tensor equations relating a stress tensor σ_{ij} and the deformation tensor g_{ij} where σ_{ij} and g_{ij} are referred to the same embedded coordinates in the body.*

Deformations or stress-strain laws which cannot be so expressed have an indefinite meaning since the deformation or stress-strain properties of the material would then depend upon the choice of coordinate system. The above principles immediately impart conceptual ease and clearness to the general precise stress-strain problem for continuous deformations. In many cases the writing of the complete equations for such a problem becomes mechanical.

Elementary examples to illustrate the procedures and equations are given in this paper. A tensor generalization of "natural strain" is found and Young's modulus and Poisson's ratio for "finite strain" are derived. Infinitesimal strain and strain velocity are defined for continuous deformations and related to the deformation tensor.

The general problem of stress and finite strain for elastic bodies has been considered by Murnaghan (3), Biot (4), and Seth (5). The author believes it would be of interest to translate the stress-strain relations assumed in these papers into tensor equations in

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[1]Unless otherwise specified all superscripts or subscripts such as i, j, k, α, β take on the values 1, 2, and 3.

[2]An expression containing a letter which is repeated as a superscript and subscript is summed on that letter unless otherwise stated.

[3]The reader is referred to Eisenhart [1][4] and McConnell [2], for accounts on tensors and their application in geometrical and physical problems.

[4]Numbers in brackets refer to the bibliography at the end of this report.

embedded coordinate systems. Definitions of finite strain based on geometrical quantities may also be so translated.

This use of a moving and deforming coordinate system may be said to be an intensification of the Lagrangian viewpoint, where the position and velocity of a particle referred to space coordinates are expressed as functions of embedded coordinates and the time t. It is closely related to the intrinsic viewpoint in geometry, where quantities such as the curvature of a surface are expressed in terms of the fundamental tensor of the surface itself without reference to the space in which the surface is immersed.

2. The general problem. Consider a body or continuous medium immersed in space. Let $\bar{x}^\alpha$, $\alpha = 1, 2, 3$, be the coordinates of points of space referred to a fixed coordinate system. Let x^i be a set of coordinates embedded in the body so that x^i are the coordinates of a particle of the body. Any deformation may be uniquely described by three displacement functions:

$$\bar{x}^\alpha = \bar{x}^\alpha(x^i, t) \tag{1}$$

which prescribe the position $\bar{x}^\alpha$ of the particle x^i at time t.

The square of the element of arc length in the space is given by a quadratic expression:

$$ds^2 = \bar{g}_{\alpha\beta}\, d\bar{x}^\alpha\, d\bar{x}^\beta,$$

where the coefficients $\bar{g}_{\alpha\beta}$ are known functions of $\bar{x}^\alpha$. Often Cartesian coordinates may be chosen so that $\bar{g}_{11} = \bar{g}_{22} = \bar{g}_{33} = 1$ and $\bar{g}_{12} = \bar{g}_{23} = \bar{g}_{31} = 0$. In the embedded coordinates

$$ds^2 = g_{ij}\, dx^i\, dx^j = \bar{g}_{\alpha\beta}\left(\frac{\partial \bar{x}^\alpha}{\partial x^i}\, dx^i\right)\left(\frac{\partial \bar{x}^\beta}{\partial x^j}\, dx^j\right).$$

Therefore g_{ij}, a function of x^i and t, is given by

$$g_{ij} = \bar{g}_{\alpha\beta}\frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial \bar{x}^\beta}{\partial x^j}, \tag{2}$$

so that g_{ij} is expressible in terms of $\bar{g}_{\alpha\beta}$ and $\bar{x}^\alpha(x^i, t)$.

We shall make use of transformation equations such as

$$\mu_i = \bar{\mu}_\alpha \frac{\partial \bar{x}^\alpha}{\partial x^i}, \qquad \lambda_{ij} = \bar{\lambda}_{\alpha\beta}\frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial \bar{x}^\beta}{\partial x^j},$$

where $\bar{\mu}_\alpha$, $\bar{\lambda}_{\alpha\beta}$ are the components of a tensor in $\bar{x}^\alpha$ coordinates and μ_i, λ_{ij} are components of the same tensor in x^i coordinates. The formulas for raising or lowering indices of tensors μ_i or λ_{ij} are formulas such as

$$\mu_i = g^{ir}\mu_r, \qquad \lambda_i^j = g^{jr}\lambda_{ri}, \qquad \lambda_{ij} = g_{ir}\lambda_j^r,$$

where g^{ir} is the "reciprocal" of the deformation tensor g_{ij}, that is,

$$g^{ir}g_{rj} = \delta_j^i,$$

where δ_j^i is the Kroneker delta:

$$\delta_j^i = 1 \quad \text{if} \quad i = j, \qquad \delta_j^i = 0 \quad \text{if} \quad i \neq j.$$

It may be noted that the six components g^{ij} may each be written in terms of the six

components g_{ij} by the rules for finding the reciprocal of a matrix. It follows that g^{ij} may also be expressed in terms of $\overline{g}_{\alpha\beta}$ and $\overline{x}^{\alpha}(x^i, t)$. Thus we may write

$$g^{ij} = g^{ij}(\overline{g}_{\alpha\beta}, \overline{x}^{\alpha}). \tag{3}$$

In accordance with the principle stated in the introduction there exists for any material a stress-strain law of form

$$\sigma_{ij} = \varphi_{ij}[g_{ij}(x^i, t), x^i, t], \tag{4}$$

where φ_{ij} is a *functional*[5] of the six functions $g_{ij}(x^i, t)$ and may vary also with the numbers x^i and t. In many cases it suffices to choose as φ_{ij} a tensor formed from g_{ij} and g^{ij} by the processes of addition, multiplication, differentiation and integration. For example φ_{ij} may be formed as the sum of such tensors as[6]

$$g_{ij}\ ,\ \int_0^t g_{ij}\,dt,\ \dot{g}_{ij}\ ,\ \int_0^t \dot{g}_{ij}\,dt,\ \int_0^t\int_0^t \dot{g}_{ij}\,dt,\ \ddot{g}_{ij}\ ,\ g^{rs}\dot{g}_{ri}\dot{g}_{sj}$$

etc., with coefficients which are functions or functionals of invariants such as

$$g^{ij}g_{ij}\ ,\ g^{ij}\int_0^t \dot{g}_{ij}\,dt,\ g^{rs}g^{up}\dot{g}_{ru}g_{sp}\ ,$$

etc. In short φ_{ij} is any covariant tensor of order two which may be formed from g_{ij} and g^{ij} by tensor addition or tensor multiplication and by differentiation or integration.[7] For a given material φ_{ij} is determined by experiment.

The equilibrium conditions may be written:

$$\sigma^i_{j,i} = \varphi^i_{j,i} = 0, \tag{5}$$

where

$$\varphi^i_{j,k} = \frac{\partial \varphi^i_j}{\partial x^k} - \varphi^i_r\begin{Bmatrix} r \\ j\ k \end{Bmatrix} + \varphi^r_j\begin{Bmatrix} i \\ r\ k \end{Bmatrix} \tag{6}$$

and

$$\begin{Bmatrix} i \\ j\ k \end{Bmatrix} = g^{ir}[jk, r],\ [jk, r] = \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r}\right) \tag{7}$$

Equations (1) to (7) combine in a straightforward manner to form three equations in the three unknown functions $\overline{x}^{\alpha} = \overline{x}^{\alpha}(x^i, t)$ which may readily be written out. Solutions of these which satisfy the boundary conditions of the problems are sought. These boundary conditions may be, for example, specified stresses $\overline{\sigma}^{\alpha\beta}$ referred to space coordinates. They may become boundary conditions on the three functions by use of the equations: $\sigma_{ij} = \varphi_{ij} = \overline{\sigma}_{\alpha\beta}\,\partial\overline{x}^{\alpha}/\partial x^i\,\partial\overline{x}^{\beta}/\partial x^j$.

3. Relationship of G_{ij} and infinitesimal strain. To translate Eqs. (2) into more familiar terms let us transform both the embedded and the space coordinates, in a given

[5] A *functional* is an association of a number with one or more functions.

[6] A dot placed over a quantity will indicate its time derivative when the x^i are kept constant.

[7] It may be readily verified that the derivative or integral of a tensor with respect to time is also a tensor with regard to any transformation of coordinates from one embedded system to another embedded system.

deformation, in such a way that both coordinate systems become Cartesian and have identical axes at the instant $t = t_1$. Then we may write

$$\bar{x}^\alpha = x^\alpha + u^\alpha,$$

where $u^i = u^i(x^j, t)$ are "displacements" and $u^i(x^j, t_1) = 0$. The one-one component of the deformation tensor is, by Eq. (2),

$$g_{11} = \left(1 + \frac{\partial u^1}{\partial x^1}\right)^2 + \left(\frac{\partial u^2}{\partial x^1}\right)^2 + \left(\frac{\partial u^3}{\partial x^1}\right)^2$$

and the one-two component is

$$g_{12} = \left(1 + \frac{\partial u^1}{\partial x^1}\right)\frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1}\left(1 + \frac{\partial u^2}{\partial x^2}\right) + \frac{\partial u^3}{\partial x^1}\frac{\partial u^3}{\partial x^2},$$

with similar expressions for the remaining components. It will be noted that these expressions are, except for a factor of two and the appearance of unity in some of the terms, identical with well known "exact" expressions for the "finite strain tensor". If t is infinitesimally near to t_1 then u^1 is infinitesimal and

$$g_{11} = 1 + 2\frac{\partial u^1}{\partial x^1}, \qquad g_{12} = \frac{\partial u^1}{\partial x^2} + \frac{\partial u^2}{\partial x^1}$$

with similar expressions for the remaining components. Now $\partial u^1/\partial x^1$, $1/2(\partial u^1/\partial x^2 + \partial u^2/\partial x^1)$ etc., are components of infinitesimal strain as usually defined[8] for the situation which exists at time $t = t_1$. The covariant components of *strain velocity* v_{ij} may be defined to be the time derivatives of these infinitesimal strains at time $t = t_1$, so that

$$\frac{1}{2}\dot{g}_{11} = v_{11} = \frac{\partial^2 u^1}{\partial x^1\,\partial t}, \qquad \frac{1}{2}\dot{g}_{12} = v_{12} = \frac{1}{2}\left(\frac{\partial^2 u^1}{\partial x^2\,\partial t} + \frac{\partial^2 u^2}{\partial x^1\,\partial t}\right), \qquad \text{etc.}$$

Alternatively, we may write

$$\frac{1}{2}\dot{g}_{11} = v_{11} = \frac{\partial v^1}{\partial x^1}, \qquad \frac{1}{2}\dot{g}_{12} = v_{12} = \frac{1}{2}\left(\frac{\partial v^1}{\partial x^2} + \frac{\partial v^2}{\partial x^1}\right), \qquad \text{etc.}$$

where $v^i = \partial u^i/\partial t$ (x_i = const) is the velocity vector (in either coordinate system) at time $t = t_1$ for the particle x^i. In tensor form, therefore, the equations for strain velocity are

$$\frac{1}{2}\dot{g}_{ij} = v_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}),$$

where $v_{i,j}$ is the covariant derivative of v_i defined by the equations

$$v_{i,j} = \frac{\partial v_i}{\partial x^j} - v_r\left\{\begin{matrix} r \\ i\ j \end{matrix}\right\},$$

$\{_i{}^r{}_j\}$ being the Christoffel symbols as defined by Eqs. (7).

Since g_{ij} , v_{ij} and $d\epsilon_{ij}$ are each tensors with regard to any change of embedded coordinates the above results do not depend upon the choice of embedded coordinates and are valid at all times t.

[8]Except, by some writers, for the factor 1/2.

The infinitesimal strain at time t for the time increment dt is simply

$$d\epsilon_{ij} = v_{ij}\,dt = \frac{1}{2}\,\dot{g}_{ij}\,dt = \frac{1}{2}\,dg_{ij}\,.$$

Thus, assuming for convenience that $g_{ij} = 0$ when $t = -\infty$,

$$\int_{-\infty}^{t} d\epsilon_{ij} = \frac{1}{2}\,g_{ij}$$

and we may say that one of the integrals of the infinitesimal strain tensor is one-half the deformation tensor in an embedded coordinate system.

By the principles stated in the introduction it was to be expected that the infinitesimal strain tensor $d\epsilon_{ij}$ should thus be expressible in terms of operations on g_{ij} since surely the infinitesimal strain tensor is independent of any rigid motion of the body.

It is instructive to consider "deformations" for which $d\epsilon_{ij} = 0$ at all times t. Such a condition may be considered to be the equivalent of a stress-strain law in a problem. If $d\epsilon_{ij} = 0$ then $v_{ij} = 1/2\ \dot{g}_{ij} = 0$ and the deformation tensor g_{ij} is a function of x^i alone and does not vary with time. Hence the arc length $ds^2 = g_{ij}\,dx^i\,dx^j$ of an embedded element dx^i is a function of x^i and dx^i alone and does not vary with time since x^i and dx^i do not vary with time. Thus the "deformation" consists of any rigid motion. The solutions of Eqs. (2) for the displacement $\bar{x}^\alpha = \bar{x}^\alpha(x^i, t)$ when $\bar{x}^\alpha$ denote Cartesian coordinates, are simply the general equations for transforming from one set of Cartesian coordinates to another such set.

Example 1. We solve a problem of a body under uniform hydrostatic pressure for the case of finite strains using a set of generalized elasticity laws. We choose as stress-strain law referred to embedded coordinates, the law:

$$\sigma_{ij} = \varphi_{ij} = A\epsilon_{ij} + B\theta g_{ij}\,, \tag{8}$$

where A and B are "elastic constants",

$$\epsilon_{ij} = \frac{1}{4}\,g_{ir}\int_0^t g^{rs}\dot{g}_{sj}\,dt + \frac{1}{4}\,g_{jr}\int_0^t g^{rs}\dot{g}_{si}\,dt \tag{9}$$

and

$$\theta = \frac{1}{2}\int_0^t g^{ij}\dot{g}_{ij}\,dt. \tag{10}$$

The tensor ϵ_{ij} as thus defined will be shown to be a generalization of "natural strain". It is easily verified that $\dot{\theta} = \dot{V}/V$, where V is an element of volume of the body. Let us choose the space coordinate $\bar{x}^\alpha$ to be Cartesian, and the embedded coordinates x^i to coincide with $\bar{x}^\alpha$ at $t = 0$. Therefore

$$\bar{g}_{11} = \bar{g}_{22} = \bar{g}_{33} = 1, \qquad \bar{g}_{12} = \bar{g}_{23} = \bar{g}_{31} = 0.$$

As trial displacement functions we take

$$\bar{x}^i = ax^i,$$

where

$$a = a(t), \qquad a(0) = 1.$$

Using Eqs. (2) the deformation tensor is found to be

$$g_{11} = g_{22} = g_{33} = a^2, \qquad g_{12} = g_{23} = g_{31} = 0.$$

The contravariant components of g_{ij} are then found to be

$$g^{11} = g^{22} = g^{33} = \frac{1}{a^2}, \qquad g^{12} = g^{23} = g^{31} = 0.$$

Equations (9) yield by direct computation

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = a^2 \log a, \qquad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0.$$

We note that raising one of the indices of ϵ_{ij} by the equation $\epsilon^i_j = g^{ir}\epsilon_{rj}$, we get

$$\epsilon^1_1 = \epsilon^2_2 = \epsilon^3_3 = \log a, \qquad \epsilon^1_2 = \epsilon^2_3 = \epsilon^3_1 = 0.$$

It may be noted that ϵ_{ij} and ϵ^i_j are considered to be different components of the same strain tensor. It turns out for this example, that ϵ^i_j and σ^i_j have identical values as the components of the strain tensor or stress tensor, respectively, calculated in experimental work. The quantity θ is found to be, for this problem: $\theta = \log a^3$. Since $\bar{x}^i = ax^i$ and $\bar{x}^i$ are Cartesian coordinates it follows that $a^3 = V/V_0$ where V is the volume of the body at time t and V_0 is the volume at time $t = 0$. Therefore $\theta = \log (V/V_0)$ or $\dot{\theta} = \dot{V}/V$.

Combining these equations we find

$$\sigma^i_j = \varphi^i_j = A\epsilon^i_j + B\theta\delta^i_j = (A + 3B)(\log a)\delta^i_j .$$

All components g_{ij} are functions of time alone and do not depend on x^1, x^2, or x^3. The Christoffel symbols $[ij, k]$ and $\{{}_i{}^s{}_k\}$ are linearly dependent on $\partial g_{ij}/\partial x^k$ and consequently vanish identically. It follows that $\varphi^i_{j,k} = \partial\varphi^i_j/\partial x^k$. Since the components are functions of time alone the quantities $\partial\varphi^i_j/\partial x^k$ vanish identically and we get $\varphi^i_{j,k} = 0$. Thus the three equilibrium conditions $\varphi^i_{j,i} = 0$ are surely satisfied.

The stress tensor at the surface of the body referred to Cartesian coordinates due to a hydrostatic pressure p is $\bar{\sigma}^\alpha_\beta = p\delta^\alpha_\beta$. Hence the mixed components of the stress tensor referred to embedded coordinates are

$$\sigma^i_j = p\delta^\alpha_\beta \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\beta}{\partial x^j} = p\delta^\alpha_\beta\left(\frac{1}{a}\,\delta^i_\alpha\right)(a\delta^\beta_j) = p\delta^i_j .$$

Thus, the boundary conditions of the problem are satisfied if we relate the parameter a to the pressure p by the equations

$$\sigma^i_j = p\delta^i_j = (A + 3B)(\log a)\delta^i_j$$

or

$$p = (A + 3B)\log a.$$

Therefore,

$$p = \frac{1}{3}(A + 3B)\log\frac{V}{V_0}.$$

This relation for pressure and volume of a body is algebraically simple because of the particular choice of definition of finite strain (9) and the stress-strain law (8). Equations of more complexity are obtained with other stress-strain laws. For example

we might have chosen $\epsilon_{ij} = 1/2\,[g_{ij}(x^i, t) - g_{ij}(x^i, 0)]$ instead of Eqs. (9). In this case

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \frac{1}{2}(a^2 - 1), \qquad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0$$

and

$$\epsilon_1^1 = \epsilon_2^2 = \epsilon_3^3 = \frac{1}{2}\left(1 - \frac{1}{a^2}\right), \qquad \epsilon_2^1 = \epsilon_3^2 = \epsilon_1^3 = 0.$$

The invariant θ is

$$\theta = \epsilon_i^i = \frac{3}{2}\left(1 - \frac{1}{a^2}\right).$$

The equilibrium conditions $\varphi^i_{i,i} = 0$ are again satisfied identically. At the surface of the material we have

$$\sigma_j^i = A\epsilon_j^i + B\theta\delta_j^i = p\delta_j^i .$$

Therefore,

$$\frac{1}{2}A\left(1 - \frac{1}{a^2}\right) + \frac{3}{2}B\left(1 - \frac{1}{a^2}\right) = p$$

or, since $a^3 = V/V_0$,

$$p = \frac{1}{2}(A + 3B)\left[1 - \left(\frac{V}{V_0}\right)^{-2/3}\right].$$

Example 2. We now apply the method to the tensile test where the stress is given by $\sigma = \sigma(t)$, $\sigma(0) = 0$. This stress is applied, let us say, in a direction parallel to the $\bar{x}^1$-axis of a set of Cartesian coordinates $\bar{x}^1, \bar{x}^2, \bar{x}^3$. Let us write for the displacement:

$$\bar{x}^1 = ax^1, \qquad \bar{x}^2 = bx^2, \qquad \bar{x}^3 = bx^3,$$

where x^1, x^2, x^3 are embedded coordinates and $a = a(t)$, $b = b(t)$ with $a(0) = 1$, $b(0) = 1$. Then,

$$ds^2 = a^2(dx^1)^2 + b^2(dx^2)^2 + b^2(dx^3)^2$$

and by inspection the deformation tensor is

$$g_{11} = a^2, \qquad g_{22} = g_{33} = b^2, \qquad g_{12} = g_{23} = g_{31} = 0.$$

The reciprocal matrix g^{ij} of g_{ij} has components

$$g^{11} = \frac{1}{a^2}, \qquad g^{22} = g^{33} = \frac{1}{b^2}, \qquad g^{12} = g^{23} = g^{31} = 0.$$

As stress-strain laws we choose again Eqs. (8), (9), and (10). The finite strains for this case are then:

$$\epsilon_{11} = a^2 \log a, \qquad \epsilon_{22} = \epsilon_{33} = b^2 \log b, \qquad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0$$

or

$$\epsilon_1^1 = \log a, \qquad \epsilon_2^2 = \epsilon_3^3 = \log b, \qquad \epsilon_2^1 = \epsilon_3^2 = \epsilon_1^3 = 0.$$

Also,

$$\theta = \frac{1}{2}\int_0^t g^{ij}g_{ij}\,dt = \log ab^2.$$

Therefore

$$\varphi_1^1 = A\log a + B\log ab^2, \qquad \varphi_2^2 = \varphi_3^3 = A\log b + B\log ab^2,$$

$$\varphi_2^1 = \varphi_3^2 = \varphi_1^3 = 0.$$

All deformation components g_{ij} are functions of time alone and do not depend on x^1, x^2, or x^3. Hence by using the same reasoning as in Example 1 it is found that the equilibrium conditions $\varphi^i_{j,i} = 0$ are satisfied. The stresses on the boundary of the specimen referred to Cartesian space coordinates are

$$\bar{\sigma}_1^1 = \sigma, \qquad \bar{\sigma}_2^2 = \bar{\sigma}_3^3 = \bar{\sigma}_2^1 = \bar{\sigma}_3^2 = \bar{\sigma}_1^3 = 0.$$

Using the transformation laws

$$\sigma_j^i = \bar{\sigma}_\beta^\alpha \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\beta}{\partial x^j},$$

we find

$$\sigma_1^1 = \sigma, \qquad \sigma_2^2 = \sigma_3^3 = \sigma_2^1 = \sigma_3^2 = \sigma_1^3 = 0$$

as the mixed components of the stress tensor on the boundary of the specimen referred to the embedded coordinates. Combining these results we find

$$\sigma = A\log a + B\log ab^2,$$

$$0 = A\log b + B\log ab^2.$$

Solving for $\log b$ by means of the second of these equations there results

$$\log b = -\frac{B}{A+2B}\log a.$$

Therefore

$$\sigma = A\frac{1+3B/A}{1+2B/A}\log a.$$

These equations are of the form of the elasticity laws for infinitesimal strains in a tensile test if we substitute for infinitesimal strains, the "finite strains" $\epsilon_1^1 = \log a$, $\epsilon_2^2 = \epsilon_3^3 = \log b$. Comparing coefficients we are led to write two possible definitions of a generalized Poisson's ratio ν. Either

$$1+\nu = \left(1+\frac{3B}{A}\right) \div \left(1+\frac{2B}{A}\right)$$

or

$$\nu = \frac{B}{A+2B}.$$

These two definitions, it conveniently turns out, are equivalent. Hence, a generalized Young's modulus is found to be

$$E = A(A + 3B) \div (A + 2B).$$

It is furthermore seen that the finite strains ϵ_1^1 , ϵ_2^2 , ϵ_3^3 , as defined by (9) are in the case of a tensile test the quantities called natural strains.

The author has attempted to obtain the exact solution of the problem of the hollow cylinder under internal pressure or pure torsion and of a parallelepiped under pure shear by the general method given here. The precise conditions on the three displacement functions are readily obtained in each case for a number of stress-strain laws but their solutions require lengthy computations. The author hopes that they may be carried out in future work in this field.

It is believed that the concepts given here may be developed further to give a general theory of continuous deformation in which the notion of energy is fundamental. The author hopes to pursue these and other questions in a later paper.

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—NOTES—

A PROBLEM OF THE TEMPERATURE DISTRIBUTION IN A MOVING MEDIUM*

By A. E. BENFIELD (*Cruft Laboratory, Harvard University*)

The thermal problem solved in this paper is primarily of geophysical interest. However, it seems worth while to give an account of it here, as the boundary conditions are rather unusual and the solution, which makes use of the Laplace method, involves a transform not given in the usual references. It is thought that the problem may be of interest to those generally concerned with questions of thermal conductivity. The geophysical implications will be discussed in another paper, to be published elsewhere.†

We wish to find the temperature $T(x, t)$, where x is the space coordinate and t is time, in a semi-infinite medium, having initially a constant temperature gradient, A, when the medium moves to the left with a positive constant speed, v, as shown in

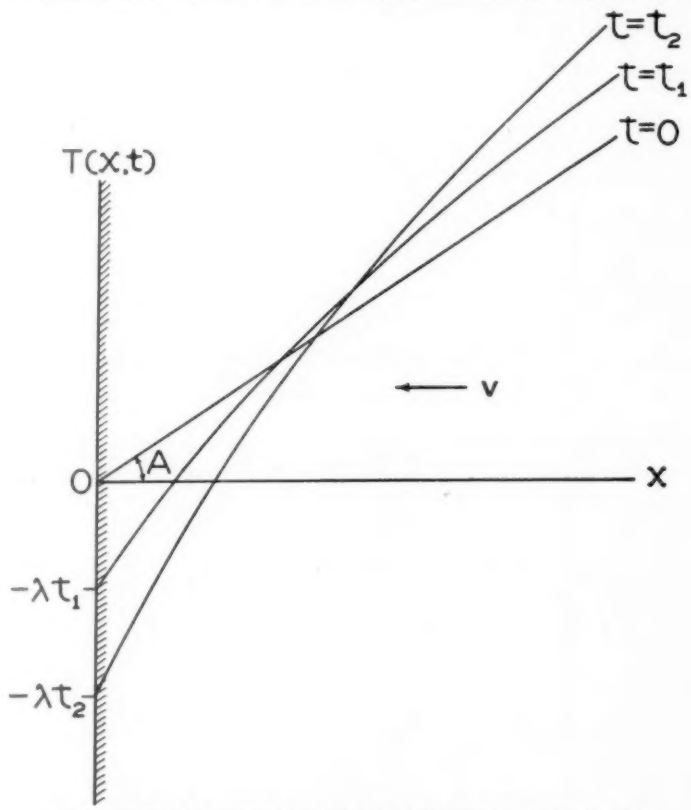

FIG. 1. Qualitative curves showing $T(x, t)$ at times $t = 0, t_1, t_2 = 2t_1$ where $0 < t_1 < t_2$. $T(0, t) = -\lambda t$.

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†A. E. Benfield, J. Appl. Phys. (in press).

Fig. 1. No material passes the boundary plane $x = 0$, but instead, on reaching it, the material is annihilated without the evolution of heat. Furthermore, the plane $x = 0$ is maintained at a temperature which decreases linearly with time at the rate λ. Therefore the boundary conditions are as follows: when $t = 0$, $T(x, 0) = Ax$; and at $x = 0$, $T(0, t) = -\lambda t$, where λ is a constant.

The application of the divergence theorem shows that the differential equation to be satisfied is

$$K \frac{\partial^2 T}{\partial x^2} + v \frac{\partial T}{\partial x} = \frac{\partial T}{\partial t} \tag{1}$$

where K is the thermal diffusivity. The medium moves in the negative x direction with this sign convention the speed $v = -dx/dt > 0$.

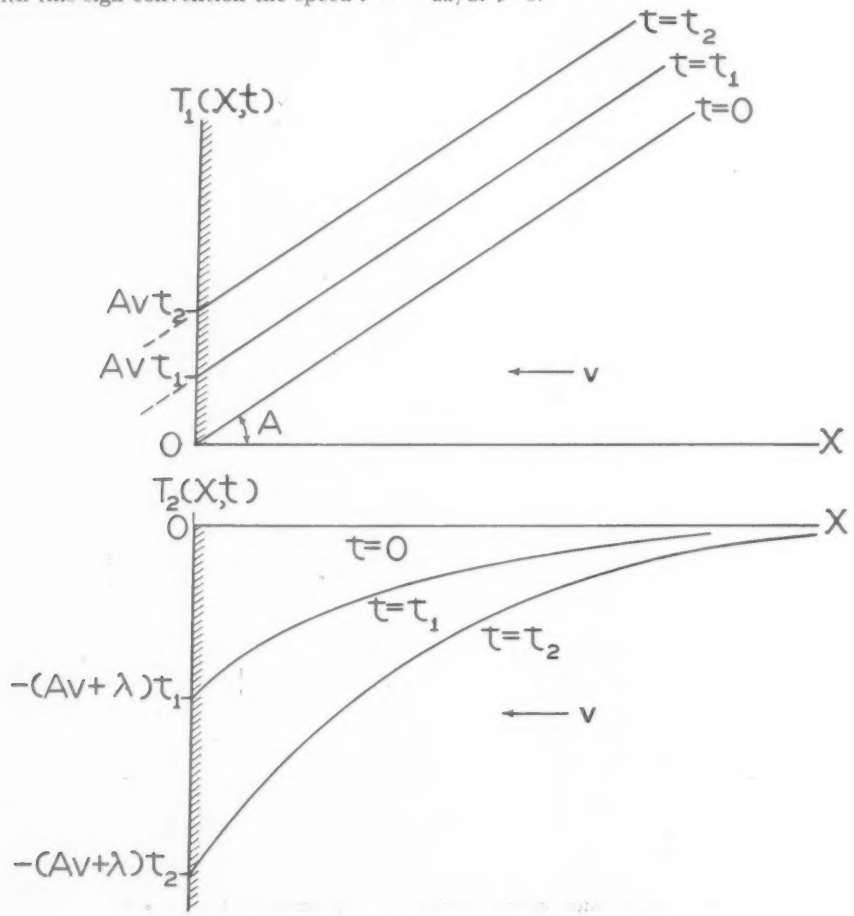

FIG. 2a. Curves for $T_1(x, t) = A(x + vt)$. $T_1(0, t) = Avt$.

FIG. 2b. Qualitative curves for $T_2(x, t)$. $T_2(0, t) = -(Av + \lambda)t$. $T(x, t) = T_1(x, t) + T_2(x, t)$.

The problem may be split into two parts; thus

$$T(x, t) = T_1(x, t) + T_2(x, t) \tag{2}$$

where

$$T_1(x, t) = A(x + vt) \tag{3}$$

as shown in Fig. 2a, and T_2, which is shown qualitatively in Fig. 2b, remains to be found. The *boundary conditions* for $T_2(x, t)$, however, are not hard to find; by considering Eq. (2), and the boundary conditions for T and T_1, we see that they must be $T_2(0, t) = -(Av + \lambda)t$ and $T_2(x, 0) = 0$. As $T_1(x, t)$ obviously satisfies Eq. (1), $T_2(x, t)$ must do the same.

Calling $\overline{T}_2$ the Laplace transform of T_2, defined by

$$\overline{T}_2 = L\{T_2\} = \int_0^\infty e^{-pt} T_2(x, t)\, dt, \tag{4}$$

Eq. (1) becomes

$$K \frac{d^2\overline{T}_2}{dx^2} + v \frac{d\overline{T}_2}{dx} - p\overline{T}_2 = 0 \tag{5}$$

where p, in Eqs. (4) and (5), is the usual constant, as used by Carslaw and Jaeger.[1]

The solution of Eq. (5), which is a second order differential equation with constant coefficients, is well-known and may be written as

$$e^{-vx/2K}[B_1 \exp\{x[v^2/4K^2 + p/K]^{1/2}\} + B_2 \exp\{-x[v^2/4K^2 + p/K]^{1/2}\}]$$

where B_1 and B_2 are the arbitrary constants, whose values depend on boundary conditions. However, as we are interested in a solution of T_2 which remains bounded, and in fact equals zero, as $x \to \infty$, we shall set $B_1 = 0$ and write

$$\overline{T}_2 = B_2 e^{-vx/2K} \exp\{-x[v^2/4K^2 + p/K]^{1/2}\}. \tag{6}$$

We must now find the value of B_2, which is done in the conventional way by considering the situation when $x = 0$, for which value of x, $\overline{T}_2 = B_2$. Since $T_2(0, t) = -(Av + \lambda)t$, it follows that when $x = 0$, $\overline{T}_2 = B_2 = -(Av + \lambda)L\{t\}$; and so, since $L\{t\} = 1/p^2$, we can now rewrite Eq. (6) as

$$\overline{T}_2 = -(Av + \lambda) e^{-vx/2K} \frac{\exp\{-x[v^2/4K^2 + p/K]^{1/2}\}}{p^2}. \tag{7}$$

In order to find T_2 we now need to know $L^{-1}[\exp\{-x[v^2/4K^2 + p/K]^{1/2}\}/p^2]$. It is known[2] that

$$L^{-1}\left[\frac{\exp\{-x[(v + p)/K]^{1/2}\}}{p}\right] = \tfrac{1}{2} \exp\{+x(v/K)^{1/2}\} \operatorname{erfc}\left[\frac{x}{2(Kt)^{1/2}} + (vt)^{1/2}\right] + \tfrac{1}{2} \exp\{-x(v/K)^{1/2}\} \operatorname{erfc}\left[\frac{x}{2(Kt)^{1/2}} - (vt)^{1/2}\right] \tag{8}$$

[1] H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, Clarendon Press, Oxford, 1947, ch. XI.
[2] Cf. Carslaw and Jaeger, *loc. cit.*, p. 270, Eqs. (2) and (3).

where $\operatorname{erfc} y = 1 - 2\pi^{-1/2} \int_0^y e^{-u^2}\, du$. If we set $\nu = v^2/4K$ and make use of the theorem[3] that

$$L\left\{\int_0^t T(t')\, dt'\right\} = \frac{1}{p} L\{T\}$$

it simply remains for us to integrate the right-hand side of Eq. (8) with respect to time. Related problems have been considered by Jaeger,[4] Paterson[5] and Horenstein.[6]

Proceeding now to the integration we find, integrating by parts, that

$$\int_0^t \operatorname{erfc} f(t)\, dt = t \operatorname{erfc} f + \frac{2}{\pi^{1/2}} \int_\infty^f t\, e^{-f^2}\, df \tag{9}$$

where $f(t) = at^{-1/2} + bt^{1/2}$, $a = x/2K^{1/2}$ and $b = \nu^{1/2} = v/2K^{1/2}$. Writing $t = (1/b)(t^{1/2}f - a)$, we find that

$$\int_\infty^f t\, e^{-f^2}\, df = \frac{a\pi^{1/2}}{2b} \operatorname{erfc} f + \frac{1}{b} \int_\infty^f t^{1/2} f\, e^{-f^2}\, df \tag{10}$$

The integral $\int_\infty^f t^{1/2} f\, e^{-f^2}\, df$ in Eq. (10) may be evaluated by integrating by parts and further substitutions, but for the case in hand this is unnecessary. This is because Eq. (8) contains not only the expression erfc f, but also erfc g where $g(t) = at^{-1/2} - bt^{1/2}$; and as a result of the symmetry of f and g it turns out that the second term of the right-hand side of Eq. (10) cancels in the integration of Eq. (8).

Therefore, combining Eqs. (9) and (10), we find that integrating Eq. (8) leads to

$$L^{-1}\left[\frac{\exp\{-x[(b^2 + p)/K]^{1/2}\}}{p^2}\right] = \tfrac{1}{2} e^{2ab}\left(t + \frac{a}{b}\right) \operatorname{erfc} f + \tfrac{1}{2} e^{-2ab}\left(t - \frac{a}{b}\right) \operatorname{erfc} g \tag{11}$$

from which it follows that

$$T_2(x, t) = -\frac{1}{2}\left(A + \frac{\lambda}{v}\right)\left[(x + vt) \operatorname{erfc}\left(\frac{x + vt}{2(Kt)^{1/2}}\right) - (x - vt)e^{-vx/K} \operatorname{erfc}\left(\frac{x - vt}{2(Kt)^{1/2}}\right)\right]. \tag{12}$$

It can be shown that Eq. (12) satisfies Eq. (1) and the boundary conditions for $T_2(x, t)$. Furthermore, one can demonstrate that if we allow $v \to \infty$, Eq. (12) leads to the appropriate expression for the temperature distribution in a stationary medium.[7]

The solution of our problem, therefore, is the sum of Eqs. (3) and (12). This is shown graphically in Fig. 1, which is the sum of Figs. 2a and 2b.

Equation (12) may be derived by other means than the method used here. For instance, one may use the substitution[8]

$$T_2 = T^* e^{-vx/2K - v^2 t/4K}$$

and make use of the fact that T^* satisfies the differential equation $K\partial^2 T^*/\partial x^2 = \partial T^*/\partial t$,

[3] Cf. Carslaw and Jaeger, *loc. cit.*, p. 242, th. IV.

[4] J. C. Jaeger, Quart. Appl. Math. 4, 100-103 (1946).

[5] S. Paterson, Quart. Appl. Math. 4, 305-306 (1946).

[6] W. Horenstein, Quart. Appl. Math. 3, 183-184 (1945).

[7] Cf. Carslaw and Jaeger, *loc. cit.*, p. 45, eq. (4).

[8] See, for instance, P. Frank and R. von Mises, *Differentialgleichungen der Physik*, F. Vieweg & Sohn, Braunschweig, 1935, vol. 2, p. 605.

thus transforming the problem into one of a stationary, rather than a moving, medium. However, on applying the new boundary conditions, for T^*, there results another untabulated Laplace transform. In this case, the required transform may be found by a differentiation, rather than an integration, but the differentiation seems to involve a step of doubtful justifiability.

In conclusion, I wish to thank Dr. P. E. LeCorbeiller for his interest in this problem and for several helpful discussions of it.

ON A CLASS OF SINGULAR INTEGRAL EQUATIONS OCCURRING IN PHYSICS*

By H. P. THIELMAN (*Iowa State College*)

1. **Introduction.** Certain boundary value problems[1] in electrodynamics can be formulated mathematically as Wiener-Hopf[2] integral equations. The equations are of the form

$$f(x) = \int_0^\infty K(|\, x - y \,|)g(y)\, dy, \qquad x > 0 \tag{1}$$

where $K(|\, x - y \,|)$ is a real function of the absolute value of the difference between x and y. The Eq. (1) is an integral equation of the "first kind", $K(|\, x - y \,|)$ and $f(x)$ are given functions, and $g(y)$ is unknown. One advantage of formulating a given problem, if possible, as a Wiener-Hopf integral equation is that such equations are susceptible to the application of the theory of Fourier transforms. The application of the latter theory is especially advantageous when the Fourier transforms of the given functions are easily found. The object of the present paper is to show that for a certain class of Wiener-Hopf integral equations a more direct method yields the explicit solution in a more elementary and simpler way. The present method applies, however, only to those integral equations of type (1), and to integral equations of the second kind such as

$$f(x) = g(x) - \lambda \int_0^\infty K(|\, x - y \,|)g(y)\, dy, \qquad x > 0 \tag{2}$$

if $K(|\, x - y \,|)$ satisfies the same linear homogeneous differential equations with constant coefficients in each of the regions $0 \leq y \leq x$, and $0 \leq x \leq y$. It should be mentioned here that the special case of Eq. (2) with $f(x) \equiv 0$, and $K(|\, x - y \,|) = e^{-|x-y|}$ was treated by Lalesco[3] by a method similar to the one given here.

2. **The general theory.** We consider an integral equation of the form

$$f(x) = \lambda \int_0^\infty K(|\, x - y \,|)g(y)\, dy \qquad x > 0 \tag{1'}$$

*Received May 11, 1948.

[1] J. F. Carlson and A. E. Heins, *The reflection of an electromagnetic plane wave by an infinite set of plates.* I, Q. Appl. Math. **4**, 313-329 (1947).

[2] N. Wiener, E. Hopf, *Über eine Klasse singulärer Integralgleichungen*, Sitzungsber. Preuss. Ak. Wissensch. 696-706 (1931).

[3] Trajan Lalesco, *Théorie des équations intégrales*, A. Hermann et Fils, Paris, 1912, p. 121.

where

a) $\int_0^\infty K(|\, x - y \,|)\, dy$ is assumed to exist,

b) $K(|\, x - y \,|)$ is continuous for all positive x and y,

c) $K(|\, x - y \,|)$ as a function of x satisfies the same linear homogeneous differential equation with constant coefficients in each of the regions $0 \leq y \leq x$, and $0 \leq x \leq y$.

Since $K(|\, x - y \,|)$ is a function of the absolute value of the difference $x - y$ only, we can write

$$K(|\, x - y \,|) = \begin{cases} F(x - y) & \text{if } 0 \leq y \leq x, \\ F(-x + y) & \text{if } 0 \leq x \leq y. \end{cases}$$

By hypothesis $F(x - y)$, and $F(-x + y)$ as functions of x satisfy the same linear homogeneous differential equation with constant coefficients. Let this be the differential equation of lowest order which they satisfy. It must then be of the form

$$\left(\sum_{i=0}^{l} a_{2i} D^{2i} \right) u = 0, \tag{A}$$

where $2l = n$, the order of the differential equation. This follows from the fact that for any m which is a root of the characteristic equation of the linear differential equation satisfied by both $F(x - y)$ and $F(-x + y)$, the negative of m is also a root. Let us write (1′) in the form

$$f(x) = \lambda \int_0^x F(x - y) g(y)\, dy + \lambda \int_x^\infty F(-x + y) g(y)\, dy. \tag{1''}$$

Taking derivatives of this function and indicating by $F'(x - y)$ the derivative with respect to the argument $|\, x - y \,|$,

$$\frac{d}{dx} F(x - y) = F'(x - y), \qquad \frac{d}{dx} F(-x + y) = -F'(-x + y),$$

we obtain

$$f'(x) = \lambda \int_0^x F'(x - y) g(y)\, dy - \lambda \int_x^\infty F'(-x + y) g(y)\, dy,$$

$$f''(x) = \lambda \int_0^x F''(x - y) g(y)\, dy + 2\lambda F'(0) g(x) + \lambda \int_x^\infty F''(-x + y) g(y)\, dy,$$

$$f'''(x) = \lambda \int_0^x F'''(x - y) g(y)\, dy + 2\lambda F'(0) g'(x) - \lambda \int_x^\infty F''(-x + y) g(y)\, dy.$$

By mathematical induction it is easily shown that

$$f^{(2i)}(x) = \lambda \int_0^x F^{(2i)}(x - y) g(y)\, dy + \lambda \int_x^\infty F^{(2i)}(-x + y) g(y)\, dy$$

$$+ 2\lambda [F^{(2i-1)}(0) g(x) + F^{(2i-2)}(0) g''(x) + \cdots + F'(0) g^{(2i-2)}(x)],$$

where $i = 1, 2, \cdots, l$.

Forming the sum

$$\sum_{i=0}^{l} a_{2i} f^{(2i)}(x)$$

we obtain

$$\sum_{i=0}^{l} a_{2i} f^{(2i)}(x) = 2\lambda \sum_{t=0}^{l-1} \sum_{i=t+1}^{l} a_{2i} F^{(2i-2t-1)}(0) g^{(2t)}(x), \tag{2'}$$

and the integrals have all cancelled each other because $F(x - y)$ and $F(-x + y)$ satisfy Eq. (A). We have thus found that $g(x)$ must satisfy a linear differential equation with constant coefficients of order $n - 2$. Hence if $n = 2$, $g(x)$ is explicitly expressed in terms of $f(x)$ and its second derivative. Equation (2') gives a necessary condition to be a solution of (1'). In order to show that it is actually a solution of (1') we substitute the general solution of (2') in (1') and find the conditions on $f(x)$ in order that $g(x)$ may exist, and that the integral in (1') may exist.

It is obvious that the theory just discussed applies not only to equations of the type (1') but also to the homogeneous equation obtained from (1') by letting $f(x) = g(x)$, the unknown function. If instead of starting with (1') we start with the integral equation of the "second kind" (2), where $K(|x - y|)$ is of the type specified above, we obtain in an analogous manner the conditions on $g(x)$ that it may be a solution of (2).

It should be noticed that the kernels of the form

$$K(|x - y|) = \sum_{i=1}^{m} e^{-k_i|x-y|} P_i(|x - y|),$$

where $k_i > 0$ and $P_i(|x - y|)$ are polynomials in $|x - y|$, are of the required type.

We shall illustrate this general theory by some examples which in themselves are of sufficient generality to warrant an independent investigation. For this reason the results obtained will be stated as theorems. The derivations of these results will illustrate the applicability of the method to problems in applied mathematics in which the reader may be interested.

3. Application of the general theory. By an application of the above theory we prove the

Theorem I. Necessary and sufficient conditions that $g(x)$ be a solution of the integral equation

$$f(x) = \lambda \int e^{-k|x-y|} g(y)\, dy \qquad x > 0,\ k > 0 \tag{B}$$

are

a) that $kf(0) - f''(0) = 0$,

b) $g(x) = \dfrac{k^2 f(x) - f''(x)}{2k\lambda}$,

c) that the integral in (B) exist, namely that $g(x)$ be of the order e^{cx} where $c < k$.

Proof. We note that the kernel satisfies the differential equation $(k^2 - D^2)u = 0$. Writing (B) in the form

$$f(x) = \lambda \int_0^x e^{-k(x-y)} g(y)\, dy + \lambda \int_x^\infty e^{-k(y-x)} g(y)\, dy \tag{B'}$$

and differentiating with respect to x twice we obtain

$$f'(x) = -\lambda k \int_0^x e^{-k(x-y)} g(y)\, dy + \lambda g(x) + \lambda k \int_x^\infty e^{-k(y-x)} g(y)\, dy - \lambda g(x),$$

$$f''(x) = \lambda k^2 \int_0^x e^{-k(x-y)} g(y)\, dy - 2\lambda g(x) + \lambda k^2 \int_x^\infty e^{-k(y-x)} g(y)\, dy.$$

Forming $k^2 f(x) - f''(x)$ we get $k^2 f(x) - f''(x) = 2\lambda k g(x)$, or

$$g(x) = \frac{k^2 f(x) - f''(x)}{2\lambda k}.$$

Substituting this value with x replaced by y into (B′) there results

$$f(x) = \frac{1}{2k} \int_0^x e^{-k(x-y)} [k^2 f(y) - f''(y)]\, dy$$

$$+ \frac{1}{2k} \int_x^\infty e^{-k(y-x)} [k^2 f(y) - f''(y)]\, dy$$

$$= \frac{k}{2} \int_0^x e^{-k(x-y)} f(y)\, dy + \frac{k}{2} \int_x^\infty e^{-k(y-x)} f(y)\, dy$$

$$- \frac{1}{2k} \int_0^x e^{-k(x-y)} f''(y)\, dy - \frac{1}{2k} \int_x^\infty e^{-k(y-x)} f''(y)\, dy.$$

Integrating the integrals involving $f''(y)$ by parts twice by letting first $dv = f''(y)\, dy$, and then $dv = f'(y)\, dy$ we get

$$f(x) = f(x) - \frac{e^{kx}}{2k} [kf(0) - f'(0)].$$

If this last equation is to be an identity $kf(0) - f'(0) = 0$, as stated in the theorem. The function $g(x)$ has to be of the order $O(e^{cx})$ in order for the integral to exist.

As a direct consequence of Theorem I we have

Theorem II. All twice differentiable solutions of the homogeneous integral equation

$$g(x) = \lambda \int_0^\infty e^{-k|x-y|} g(y)\, dy \tag{C}$$

are

a) $g(x) = c_1(k^{1/2} \sin \{[k(2\lambda - k)]^{1/2} x\} + (2\lambda - k)^{1/2} \cos \{[k(2\lambda - k)]^{1/2} x\})$

if $\lambda > \frac{k}{2}$,

b) $g(x) = c_2(1 + kx)$ if $\lambda = \frac{k}{2}$,

c) $g(x) = c_3(k^{1/2} \sinh [k(k - 2\lambda)^{1/2}]x + (k - 2\lambda)^{1/2} \cosh [k(k - 2\lambda)^{1/2}]x)$

if $0 < \lambda < k/2$, where the $c_i (i = 1, 2, 3)$ are arbitrary constants.

Proof. By setting $f(x)$ in Theorem I equal to $g(x)$ we obtain $kg(0) - g'(0) = 0$, $g(x) = k^2g(x) - g''(x)/2\lambda k$. Rewriting the last equation for $g(x)$ we obtain

$$g''(x) + k(2\lambda - k)g(x) = 0.$$

We have now three possible cases: a) $2\lambda - k > 0$, b) $2\lambda - k = 0$, and c) $2\lambda - k < 0$.

For case a) the general solution of the differential equation satisfied by $g(x)$ is

$$g(x) = A \sin [k(2\lambda - k)]^{1/2}x + B \cos [k(2\lambda - k)]^{1/2}x.$$

Imposing the condition a) of Theorem I on $g(x)$, i.e. $kg(0) - g'(0) = 0$, we find that $B/A = [(2\lambda - k)/k]^{1/2}$, or

$$g(x) = c_1(k^{1/2} \sin [k(2\lambda - k)]^{1/2}x + (2\lambda - k)^{1/2} \cos [k(2\lambda - k)]^{1/2}x).$$

For case b), when $2\lambda = k$, $g''(x) = 0$, and $g(x) = c_1(1 + kx)$ where c_2 is an arbitrary constant.

For case c) $2\lambda - k < 0$, and

$$g(x) = A \sinh [k(k - 2\lambda)]^{1/2}x + B \cosh [k(k - 2\lambda)]^{1/2}x.$$

Since, however, $kg(0) - g'(0) = 0$,

$$g(x) = c_3(k^{1/2} \sinh [k(k - 2\lambda)]^{1/2}x + (k - 2\lambda)^{1/2} \cosh [k(k - 2\lambda)]^{1/2}x).$$

$g(x)$ has to be of order $O(e^{cx})$ with $c < k$, hence $[k(k - 2\lambda)]^{1/2} < k$ or $\lambda > 0$.

We see that the characteristic values, λ, constitute a continuous spectrum, a phenomenon which is due entirely to the fact that our integral equation has an unbounded region of integration. For integral equations with bounded kernels, and bounded regions of integration it is well known that the number of characteristic values is at most denumerable. If in Theorem II we set $k = 1$, we obtain the example treated by Lalesco[3].

By direct application of the general theory we obtain *Theorem III.* If $g(x)$ is a solution of the integral equation

$$f(x) = \lambda \int_0^\infty e^{-k|x-y|} \mid x - y \mid g(y)\, dy \tag{D}$$

then $g(x)$ must satisfy the differential equation

$$g''(x) + k^2g(x) = \frac{f^{iv}(x) - 2k^2f''(x) + k^4f(x)}{2\lambda}$$

Proof. The kernel satisfies the differential equation

$$(D^4 - 2k^2D + k^4)u(x) = 0,$$

in each of the regions $0 \leq y \leq x$, and $0 \leq x \leq y$. Writing (D) in the form

$$f(x) = \lambda \int_0^x e^{-k(x-y)}(x - y)g(y)\, dy + \lambda \int_x^\infty e^{-k(y-x)}(y - x)g(y)\, dy,$$

differentiating four times and forming the expression on the left of the following equation we obtain,

$$f^{iv}(x) - 2k^2f''(x) + k^4f(x) = 2\lambda k^2g(x) + 2\lambda g''(x),$$

which can be written as

$$g''(x) + k^2 g(x) = \frac{f^{iv}(x) - 2k^2 f''(x) + k^4 f(x)}{2\lambda}.$$

If we set $f(x) \equiv g(x)$ in (D) we obtain the *Theorem IV*. If $g(x)$ is a solution of the homogeneous equation

$$g(x) = \lambda \int_0^\infty e^{-k|x-y|} \mid x - y \mid g(y)\, dy \qquad x > 0, k > 0 \tag{E}$$

then $g(x)$ must satisfy the differential equation

$$g^{iv}(x) - 2(\lambda + k^2)g''(x) - k^2(2\lambda - k^2)g(x) = 0.$$

The real parts of the roots of the characteristic equation of the last differential equation have to be in absolute value less than k in order that the integral may exist with $g(x)$ a solution of (E). The complete discussion of the spectrum of characteristic values of this equation, as well as the treatment of certain integral equations of the second kind is reserved for another paper.

ON THE GENERAL EQUATIONS OF PROBLEMS OF AXIAL SYMMETRY IN THE THEORY OF PLASTICITY*

By P. S. SYMONDS (*Brown University*)

1. Introduction. The solutions of axially symmetric problems in plasticity which have been presented in the literature depend in general upon the assumption of the so-called "full plasticity" condition. According to this assumption, two principal stress differences are taken equal to the yield stress in pure tension. This implies that two of the principal stresses are equal, and thus reduces the problem to a statically determinate one. On this basis solutions have been given for a number of important problems, such as those of the indentation of a rigid surface of revolution into a plastic mass (see for example papers of Hencky[1] and Ishlinsky[2]).

This is a quite fictitious yield condition, with little physical or mathematical justification. Hence it is of interest to examine the general equations for axial symmetry using the well confirmed Mises flow condition, and, in particular, to determine whether or not these equations form a system of hyperbolic type. If so, the techniques of numerical integration using the networks of characteristic curves, which have been developed for supersonic fluid flow problems, could be applied to these problems of plasticity. Unfortunately the answer to this question, as shown in the present note, is that the equations are actually of elliptic type, the "characteristics" being real only under the special condition which reduces the problem to one of plane strain.

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[1]H. Hencky, *Über einige statisch bestimmte Fälle des Gleichgewichts in plastischen Körpern*, Z. angew. Math. Mech. 3, 241-251 (1923).

[2]A. Ishlinsky, *The problem of plasticity with axial symmetry and Brinell's test*, Prikl. Mat. Mekh. 8, 201-224 (1944).

2. The general equations. In problems of axial symmetry defined by an axial coordinate z and a radial coordinate r we are concerned with four stress components σ_z, σ_r, σ_θ, τ_{rz}, and with two velocity components u, w. The two equations of equilibrium are

$$\left.\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0, \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} &= 0. \end{aligned}\right\} \tag{1}$$

The Mises yield condition becomes

$$(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6\tau_{rz}^2 = 6K^2, \tag{2}$$

where K is the yield stress in pure shear. The problem is evidently not statically determinate, so we must add the stress-strain relations to the foregoing. According to the Saint Venant-Levy-Mises flow theory[3]

$$\left.\begin{aligned} e_r = \frac{\partial u}{\partial r} &= \frac{1}{3}\lambda(2\sigma_r - \sigma_\theta - \sigma_z), \\ e_\theta = \frac{u}{r} &= \frac{1}{3}\lambda(2\sigma_\theta - \sigma_z - \sigma_r), \\ e_z = \frac{\partial w}{\partial z} &= \frac{1}{3}\lambda(2\sigma_z - \sigma_r - \sigma_\theta), \\ g_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) &= \lambda\tau_{rz}, \end{aligned}\right\} \tag{3}$$

where e_r, e_θ, e_z, g_{rz} are (tensor) strain-rate components (taken with respect to an arbitrary base, not necessarily time), and $\lambda(r, z)$ is a scalar point function depending on the material and on the state of stress at a generic point. We then have seven equations for the seven unknowns $(\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}, \lambda, u, w)$.

Further work is facilitated by changing notation as follows

$$\left.\begin{aligned} \sigma_r &= \omega + \xi, \\ \sigma_\theta &= \omega + \eta, \\ \sigma_z &= \omega + \zeta = \omega - (\xi + \eta), \\ \tau_{rz} &= \tau, \end{aligned}\right\} \tag{4}$$

where $\omega \equiv (\sigma_r + \sigma_\theta + \sigma_z)/3$, and ξ, η, ζ are the "stress deviation" components. In the following, all subscripts indicate partial differentiation. The equations become:

[3] See for the instance William Prager, *The stress-strain laws of the mathematical theory of plasticity—a survey of recent progress*, J. Appl. Mech. 15, 226-234 (1948).

$$\left.\begin{array}{ll} \text{Equilibrium condition} & \begin{cases} \omega_r + \xi_r + \tau_z = \dfrac{1}{r}(\eta - \xi), & \text{(a)} \\ \omega_z - \xi_z - \eta_z + \tau_r = -\dfrac{1}{r}\tau, & \text{(b)} \end{cases} \\ \text{Yield conditions} & \{\xi^2 + \eta^2 + \xi\eta + \tau^2 = K^2; \quad \text{(c)} \end{array}\right\} \tag{5}$$

$$\left.\begin{array}{ll} \text{Stress-strain relations} & \begin{cases} u_r = \lambda\xi, & \text{(a)} \\ u = r\lambda\eta, & \text{(b)} \\ w_z = -\lambda(\xi + \eta), & \text{(c)} \\ u_z + w_r = 2\lambda\tau. & \text{(d)} \end{cases} \end{array}\right\} \tag{6}$$

The four stress deviation components and λ are determined by the five equations consisting of Eqs. (5) and two equations of compatibility derived from Eqs. (6). The compatibility equations may be written in the form

$$\left.\begin{aligned} (\lambda\xi)_{zz} &= 2(\lambda\tau)_{rz} + (\lambda\xi + \lambda\eta)_{rr}, \\ r(\lambda\eta)_{zz} &= 2(\lambda\tau)_z + (\lambda\xi + \lambda\eta)_r \end{aligned}\right\} \tag{7}$$

as may be verified by substitution in (6).

3. Investigation of characteristic curves. We now investigate the possibility of the existence of characteristic curves C, on which

$$z = z(r) \tag{8}$$

such that on C the second derivatives of the unknowns (ω, ξ, η, τ, λ) are undefined although the functions themselves and all their first derivatives are defined. We first eliminate $\omega = (\sigma_r + \sigma_\theta + \sigma_z)/3$ between Eqs. (5a) and (5b), obtaining

$$2\xi_{rz} + \eta_{rz} - \tau_{rr} + \tau_{zz} = \frac{1}{r}\left(\eta_z - \xi_z + \tau_r - \frac{1}{r}\tau\right). \tag{9a}$$

We next differentiate Eq. (5c) twice with respect to z, obtaining

$$(2\xi + \eta)\xi_{zz} + (\xi + 2\eta)\eta_{zz} + 2\tau\tau_{zz} = -2(\xi_z^2 + \xi_z\eta_z + \eta_z^2 + \tau_z^2). \tag{9b}$$

The justification for this is that we are investigating possible discontinuities in second derivatives, hence we are interested only in how Eq. (5c) defines the relations between second derivatives. Finally, we write out Eqs. (7) and obtain equations on the left sides of which only second derivatives and the functions themselves appear:

$$\begin{aligned} \lambda\xi_{rr} - \lambda\xi_{zz} + \lambda\eta_{rr} + 2\lambda\tau_{rz} + (\xi + \eta)\lambda_{rr} + 2\tau\lambda_{rz} - \xi\lambda_{zz} \\ = -2(\lambda_r\xi_r - \lambda_z\xi_z + \lambda_r\eta_r + \lambda_r\tau_z + \lambda_z\tau_r), \end{aligned} \tag{9c}$$

$$\lambda\eta_{zz} + \eta\lambda_{zz} = -2\lambda_z\eta_z + \frac{1}{r}[2(\lambda\tau)_z + (\lambda\xi + \lambda\eta)_r]. \tag{9d}$$

Noting that a function $\varphi(r, z)$ given on a curve C [defined as $z = z(r)$] is a function of r only, on C we have

$$\frac{d\varphi}{dr} = \varphi_r + P\varphi_z \; ; \qquad \varphi_r = \frac{d\varphi}{dr} - P\varphi_z$$

where $P = dz/dr$. In particular if $\varphi = \xi_z$ or ξ_r we obtain, respectively:

$$\xi_{zr} = \frac{d\xi_z}{dr} - P\xi_{zz} \, ,$$

$$\xi_{rr} = \frac{d\xi_r}{dr} - P\frac{d\xi_z}{dr} + P^2\xi_{zz} \, .$$

In like manner the second derivatives of all the functions appearing in Eqs. (9) may be written in terms of the double derivatives with respect to z, and the parameter $P = dz/dr$ of the curve C. Equations (9) then take the form

$$\left.\begin{aligned} &A\xi_{zz} + B\eta_{zz} + C\tau_{zz} + D\lambda_{zz} = K, \\ &A'\xi_{zz} + B'\eta_{zz} + C'\tau_{zz} + D'\lambda_{zz} = K', \\ &A''\xi_{zz} + B''\eta_{zz} + C''\tau_{zz} + D''\lambda_{zz} = K'', \\ &A'''\xi_{zz} + B'''\eta_{zz} + C'''\tau_{zz} + D'''\lambda_{zz} = K''', \end{aligned}\right\} \tag{10}$$

where $A, B, \cdots, D'''$ contain only the functions and the parameter P. The "characteristic condition" of the system is thus the equation:

$$\begin{vmatrix} A & B & C & D \\ A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \\ A''' & B''' & C''' & D''' \end{vmatrix} = 0. \tag{11}$$

Determining the coefficients $A, B, \cdots, D'''$ for the system (9), we obtain the result:

$$\begin{vmatrix} -2P & -P & -P^2 + 1 & 0 \\ 2\xi + \eta & \xi + 2\eta & 2\tau & 0 \\ \lambda(P^2 - 1) & \lambda P^2 & -2\lambda P & (\xi + \eta)P^2 - 2\tau P - \xi \\ 0 & \lambda & 0 & \eta \end{vmatrix} = 0.$$

This is a fourth degree algebraic equation in P, which reduces to

$$[(P^2 - 1)(2\xi + \eta) - 4\tau P]^2 = -3[\eta(P^2 + 1)]^2.$$

The four roots are

$$P = \frac{dz}{dr} = \frac{\tau \pm K}{\xi + \eta(1 \pm i3^{1/2})/2} \tag{12}$$

where the $\pm$ signs are independent; the yield equation in the original form of Eq. (5c) has been used in obtaining (12).

4. Conclusion. It is clear that real characteristics do not exist in general, since P can be real only when $\eta = \sigma_\theta - \omega = 0$. But when $\eta = 0$ and $u \neq 0$ the stress equations become identical with the three equations for plane strain, under the Mises yield condition; the characteristic parameter P then reduces to

$$P = \frac{\tau \pm K}{\xi}.$$

As may be easily verified, this is the standard result for the slope of the characteristic curves in problems of plane plastic flow[4], which provides a useful check on the present results.

An analogous but simpler investigation of Eqs. (6) leads to the conclusion that there are also no curves on which the velocity derivatives are undefined, even though the values of u, w and the stresses are defined.

[4]See, for example, William Prager, *Plasticity for the aerodynamicist*, J. Aero. Sci. 15, 253-262 (1948).

EFFECT OF HYPERBOLIC NOTCHES ON THE STRESS DISTRIBUTION IN A WOOD PLATE*

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1. Introduction. If from an orthotropic material a flat plate is cut parallel to a plane of elastic symmetry, it will have two perpendicular axes of symmetry in the plane of the plate. Such a plate is said to be orthotropic.[1] An example of this kind of plate is a plain-sawn board. In the discussion that follows, an orthotropic plate is assumed to be bounded on two sides by hyperbolic notches given by the equation $y^2/a^2 - x^2/b^2 = 1$, and indefinitely extended in the other direction. The x- and y-axes are taken parallel to the axes of symmetry. The plate is subjected only to forces directed parallel to the x-axis and applied in the plane of the plate. These forces are assumed to act at great distances from the y-axis, and in such a way that the traction over any cross-section perpendicular to the x-axis is statically equivalent to a single force of magnitude P directed along the x-axis. The problem will be treated as one of plane stress.

2. The stress distribution in a wood plate with hyperbolic notches. The components of stress and strain in the orthotropic plate described are connected by the following relations[2]:

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[1]A. T. Price, Phil. Trans. (A) 228, 1-62 (1928). H. W. March, Physics, 7, 32-41 (1936). U. S. Forest Products Laboratory Reports Nos. 1300, 1301, 1304, 1312, 1316.

[2]U. S. Forest Products Laboratory Report No. 1503. Love's notations for the stress and strain components are used, (A. E. H. Love, *The mathematical theory of elasticity*, 4th ed., Cambridge University Press, Cambridge, 1927).

$$e_{xx} = \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y ,$$

$$e_{yy} = -\frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y , \tag{1}$$

$$e_{xy} = \frac{1}{\mu_{xy}} X_y .$$

In these equations E_x and E_y are Young's moduli in the x- and y-directions, respectively. Poisson's ratio σ_{xy} is the ratio of the contraction parallel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-axis and similarly for σ_{yx}. The quantity μ_{xy} is the modulus of rigidity associated with the directions of x and y.

Since the equations of equilibrium are deduced with no reference to the law connecting stresses and strains, they will be satisfied by a stress function F such that

$$X_x = \frac{\partial^2 F}{\partial y^2}, \qquad Y_y = \frac{\partial^2 F}{\partial x^2}, \qquad X_y = -\frac{\partial^2 F}{\partial x\, \partial y}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and then making use of the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x\, \partial y},$$

we obtain the following differential equation, satisfied by the stress function F[3]:

$$\frac{\partial^4 F}{\partial x^4} + 2K \frac{\partial^4 F}{\partial x^2\, \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \tag{3}$$

where[4]

$$K = \frac{(E_x E_y)^{1/2}}{2}\left(\frac{1}{\mu_{xy}} - \frac{2\sigma_{xy}}{E_x}\right) \tag{4}$$

and

$$\eta = \epsilon y \tag{5}$$

with

$$\epsilon = (E_x/E_y)^{1/4}. \tag{6}$$

The solution of Eq. (3) is facilitated by taking F to be of the form

$$F = R[F_1(x + i\alpha\eta) + F_2(x + i\beta\eta)] \tag{7}$$

where

$$\alpha = [K + (K^2 - 1)^{1/2}]^{1/2}, \qquad \beta = [K - (K^2 - 1)^{1/2}]^{1/2}, \tag{8}$$

i is the imaginary unit, and the letter R means that the real part of the expression in the bracket is to be taken. The functions F_1 and F_2 are any analytic functions of the complex variables $x + i\alpha\eta$ and $x + i\beta\eta$, respectively.

[3]For an orthotropic plate like the one discussed here the relation $\sigma_{xy}E_y = \sigma_{yx}E_x$ holds. See U. S. Forest Products Laboratory Report No. 1503, p. 7.

[4]For wood, K as defined here is probably always greater than 1.

The stress distribution in an orthotropic plate in a state of plane stress is then obtained by finding a solution to Eq. (3) in the form of Eq. (7), and in addition requiring that the function F satisfies the following relations on the boundary of the plate[5]:

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) = X_\nu\,, \qquad \frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) = -Y_\nu$$

where ds denotes an element of the boundary, and $X_\nu\, ds$ and $Y_\nu\, ds$ represent the x- and y-components of the force per unit thickness of the plate acting on the element of arc ds from the positive side as indicated by the direction of ν, the normal to the boundary.

A suitable choice of F for the problem considered is

$$F = R[pB\{z_1 \log (z_1 + w_1) - w_1 - z_2 \log (z_2 + w_2) + w_2\}] \tag{9}$$

where

$$z_1 = x + i\alpha\eta, \qquad z_2 = x + i\beta\eta,$$

$$w_1 = (z_1^2 + \gamma_1^2)^{1/2}, \qquad w_2 = (z_2^2 + \gamma_2^2)^{1/2},$$

$$\gamma_1^2 = b^2 + \alpha^2\epsilon^2 a^2, \qquad \gamma_2^2 = b^2 + \beta^2\epsilon^2 a^2,$$

$$B = \frac{a}{\epsilon\{\beta \arctan (\beta\epsilon\lambda) - \alpha \arctan (\alpha\epsilon\lambda)\}}, \qquad \lambda = \frac{a}{b}, \tag{10}$$

and $p = P/2ad$, the average stress on the cross-section $x = 0$, since the plate is taken to be of thickness d. To make the functions appearing in Eq. (9) single valued the following relations must be assumed always to hold:

$$R[w_1] > 0, \qquad R[w_2] > 0,$$

$$-\frac{\pi}{2} < \operatorname{amp} (z_1 + w_1) < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < \operatorname{amp} (z_2 + w_2) < \frac{\pi}{2} \tag{11}$$

By means of Eqs. (2) and (9) the stress components are found to be

$$\frac{\partial^2 F}{\partial x^2} = Y_y = R\left[pB\left(\frac{1}{w_1} - \frac{1}{w_2}\right)\right], \tag{12}$$

$$\frac{\partial^2 F}{\partial y^2} = X_x = R\left[pB\left(\frac{-\alpha^2\epsilon^2}{w_1} + \frac{\beta^2\epsilon^2}{w_2}\right)\right], \tag{13}$$

$$-\frac{\partial^2 F}{\partial x\, \partial y} = X_y = R\left[pB\left(\frac{-i\alpha\epsilon}{w_1} + \frac{i\beta\epsilon}{w_2}\right)\right]. \tag{14}$$

The stress component X_x as given by Eq. (13) was calculated at several points along the y-axis for a plain-sawn plate of Sitka spruce with hyperbolic notches, as in

[5] S. Timoshenko, *Theory of elasticity*, McGraw-Hill Co., New York, 1934, p. 179. U. S. Forest Products Laboratory Report No. 1510, p. 7.

the problem discussed here. The x-axis was taken parallel to the grain of the wood. The elastic constants were taken to be:

$$E_x = 1.679 \times 10^6 \text{ lb./in.}^2, \qquad \mu_{xy} = 0.112 \times 10^6 \text{ lb./in.}^2,$$

$$E_y = 0.076 \times 10^6 \text{ lb./in.}^2, \qquad \sigma_{xy} = 0.464$$

The results of these calculations for several values of λ are given in the table below.

VALUES OF X_x FROM EQ. (13)

$\lambda = 10$		$\lambda = 1.0$		$\lambda = 0.1$	
y	X_x/p	y	X_x/p	y	X_x/p
$0.0a$	0.6373	$0.0a$	0.6989	$0.0a$	0.9784
$0.2a$	0.6505	$0.2a$	0.7142	$0.2a$	0.9808
$0.4a$	0.6955	$0.4a$	0.7662	$0.4a$	0.9884
$0.6a$	0.7971	$0.6a$	0.8832	$0.6a$	1.001
$0.8a$	1.064	$0.8a$	1.180	$0.8a$	1.020
$1.0a$	30.84	$1.0a$	3.192	$1.0a$	1.046

It is evident that the value of the stress component X_x at $x = 0$, $y = a$ is the maximum stress occurring in the plate in each case. Setting $x = 0$, $y = a$ in Eq. (13), it is found that

$$\frac{(X_x)_{\max}}{p} = \frac{\epsilon^2(\beta^2 - \alpha^2)}{b} B, \tag{15}$$

which gives the ratio of the maximum stress to p, the average stress on the cross-section $x = 0$.

3. Isotropic stress function as the limiting case of the orthotropic stress function. It is of interest to obtain from Eq. (9) the stress function for the corresponding problem for an isotropic material. For an isotropic material $\alpha = \beta = \epsilon = 1$, as is shown by Eqs. (4), (6), and (8). For these values, the right side of Eq. (9) becomes indeterminate. Evaluating, it yields

$$F = R\left[\frac{(1 + \lambda^2)ap}{(1 + \lambda^2) \arctan \lambda + \lambda}\left\{\frac{a^2}{z + w} - iy \log (z + w)\right\}\right] \tag{16}$$

where

$$z = x + iy,$$
$$w = (z^2 + \gamma^2)^{1/2}, \qquad \gamma^2 = a^2 + b^2. \tag{17}$$

To make F single-valued, relations similar to those of Eq. (11) must hold. By letting $z = \gamma z' = \gamma(x' + iy')$ in Eq. (16), it is found that

$$Y_y = \frac{\partial^2 F}{\partial x^2} = \frac{1}{\gamma^2}\frac{\partial^2 F}{\partial x'^2} = \frac{\partial^2}{\partial x'^2}\left(\frac{F}{\gamma^2}\right) \tag{18}$$

with similar expressions for the other stress components. If, therefore, in Eq. (16) we replace z by γz and then divide the right side by γ^2, the resulting expression can be used as the stress function. Hence the change to elliptic coordinates, which is usually done by replacing z by $\gamma \sinh (u + iv)$, can be accomplished in Eq. (16) by substituting $\sinh (u + iv)$ for z and then dividing the right side by γ^2. Thus we find the following expression for the stress function in elliptic coordinates:

$$F = p \frac{\sin v_0}{v_0 + \sin v_0 \cos v_0} [yv + (\sin^2 v_0)e^{-u} \cos v] \tag{19}$$

where the hyperbola $y^2/a^2 - x^2/b^2 = 1$ is given by the particular value, v_0, of the coordinate v. Equation (19) is the stress function for the corresponding problem involving the isotropic plate.[6]

[6]See paper by H. Neuber, Zeitsch. f. Angew. Math. u. Mech. **13**, 439 (1933).

BOOK REVIEWS

Eigenfunction expansions associated with second-order differential equations. By E. C. Titchmarsh. Oxford, at the Clarendon Press, 1946. 175 pp. $7.00.

The subject matter of this book is composed of the analysis required to develop and justify the expansion of an arbitrary function $f(x)$ in the eigenfunctions of the differential equation $y'' + [\lambda - q(x)]y = 0$. The classical theory is disposed of in Chapter 1 and the remainder of the book is largely devoted to those situations where $q(x)$ is singular at an end-point of the region and/or those where $f(x)$ is defined over an infinite or semi-infinite domain. In Chapter 4 one finds as examples the expansion formulae for Hermite, Bessel, Legendre, Sonine, Laguerre, and hypergeometric functions. The latter chapters deal with the distribution of the eigenvalues, the nature of the spectrum, and with questions of convergence and summability.

The book is intended primarily for the mathematician, but its interest and value to the physicist and engineer should be great.

G. F. Carrier

Modern operational calculus with applications in technical mathematics. By N. W. McLachlan. MacMillan and Co., Ltd., London, 1948. xiv + 218 pp. $5.00.

This introduction to the theory of the Laplace Transform is intended as a guide to those interested in the application of this transform to engineering problems. However, the questions of rigor are treated in more detail than is customary in texts with this purpose. In fact, nearly one third of the book consists of appendices dealing with such items as convergence questions. In Chapters 1 and 2 the transform is defined and the fundamental theorems and identities are deduced. In Chapters 3 and 4, the solution by transform methods of ordinary and partial linear differential equations is discussed. These and the subsequent chapters which, for the most part, are concerned with integral evaluation contain many examples drawn largely from electric and acoustic problems. It is regrettable that no reference has been made to the role of this transform in solving stability problems.

The book should be especially useful to engineers whose background is essentially non-mathematical since the points of rigor discussed appear to have been selected for just such an audience.

G. F. Carrier

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